STOR 634 Review (F12)

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Contents

1 Rings and fields

1.1 Set algebra basics

Theorem 1.1. (Some basic set inequalities)

1. $E - (F \cup G) = (E - F) \cap (E - G)$
2. $E - (F \cap G) = (E - F) \cup (E - G)$
3. $E \cap (F - G) = (E \cap F) - (E \cap G)$
4. $(E - F) \cap (G - H) = (E \cap G) - (F \cup H)$

Theorem 1.2. (Some symmetric difference properties)

1. **(Important):** $E\Delta F = (E \cup F) - (E \cap F)$
2. $E\Delta(F\Delta G) = (E\Delta F)\Delta G$
3. $E \cap (F\Delta G) = (E \cap F)\Delta(E \cap G)$
4. $E\Delta F = G\Delta H \;\Rightarrow\; E\Delta G = F\Delta H$

Theorem 1.3. $\liminf E_n \subset \limsup E_n$

Proof. Note that, if $x \in \liminf E_n$, then $x \in \cap_{m=N}^{\infty} E_m$ for some N. Thus $x \in \cup_{m=n}^{\infty}$ for any n.

□

Theorem 1.4. A monotone increasing (decreasing) sequence $\{E_n\}_{n\geq 1}$ is convergent and $\lim E_n = \cup_1^{\infty} E_n$ ($\cap_1^{\infty} E_n$).

Proof. Note that, if $E_n \uparrow$, then $\limsup E_n = \cap_{n=1}^{\infty} \cup_{m=1}^{\infty} E_m$. For the lower limit, note that if $E_n \uparrow$, then $\cap_{m=n}^{\infty} E_m = E_n$.

1.2 Rings, fields, σ-rings, σ-fields

Theorem 1.5. (Properties of rings)

1. Rings contain $\emptyset$
2. Rings are closed under symmetric differences and intersections

   Proof. By definition of Δ and by $E \cap F = (E \cup F) - (E\Delta F)$.

Theorem 1.6. (Sufficient conditions for rings)

If a nonempty class of sets $\mathcal{R}$ is closed under either:

1. unions and proper differences
2. intersections, proper differences, and disjoint unions

Then it is a ring.

Proof. For (1), note that $E - F = (E \cup F) - F$. For (2), note

$$E \cup F = [E - (E \cap F)] \cup [F - (E \cap F)] \cup (E \cap F)$$

□

Theorem 1.7. (Sequences in rings)

Let $\{E_n\}_{n \geq 1}$ be a sequence of sets in a ring $\mathcal{R}$ and let $E = \cup_1^\infty E_n$ (not necessarily in $\mathcal{R}$). Then

1. $E = \cup_1^\infty F_n = \lim F_n$, where $F_n = \cup_{i=1}^n E_i$ (increasing)
2. $E = \cup_1^\infty G_n$ where G_n are disjoint sets of $\mathcal{R}$ such that $G_n \subset E_n$

Proof. (1) is trivial. For (2), write $F_n = \cup_{m=1}^n E_m$ and define $\{G_n\}$ by $G_1 = F_1$ and $G_n = F_n - F_{n-1}$.

□

Theorem 1.8. (Relation between (σ-)rings and (σ-)fields)

A (σ-)field is a (σ-)ring of which the whole space X is a member, and vice versa.

Remark. Adding X to a ring will **not** necessarily make it a field.

Proof. To prove one direction, use the definition $E - F = E \cap F^c$. The other direction is trivial.

To prove the result for σ-rings and σ-fields, use the result for rings and fields and then check closure under countable union.

□

1.3 Semirings, generated rings and fields

Lemma 1.9. Let $\mathcal{R}_\gamma$ be a (σ-)ring(field) for each γ in some arbitrary index set Γ. Define $\mathcal{R} = \cap_\gamma \{\mathcal{R}_\gamma : \gamma \in \Gamma\}$. Then $\mathcal{R}$ is a (σ-)ring(field).

Proof. Apply definition of $\cap$. The same argument applies for fields, σ-rings, and σ-fields.

□

Theorem 1.10. (Existence of generated rings/fields)

Let $\mathcal{E}$ be any class of sets. Then there exists a unique (σ-)ring(field) $\mathcal{R}_0$ such that $\mathcal{R}_0 \supset \mathcal{E}$ and, if $\mathcal{R}$ is any other (σ-)ring(field) containing $\mathcal{E}$, then $\mathcal{R} \supset \mathcal{R}_0$ (denoted $\mathcal{R}(\mathcal{E})$).

Proof. Let $\mathcal{R}_\gamma$ be any ring containing $\mathcal{E}$ and let Γ index all such rings. Consider the set $\mathcal{R}_0 = \cap_{\gamma \in \Gamma} \mathcal{R}_\gamma$.

The same argument applies for fields, σ-rings, and σ-fields.

□

Theorem 1.11. If $\mathcal{E}$ is any non-empty class of sets, any set in $\mathcal{R}(\mathcal{E})$ can be covered by a finite union of sets in $\mathcal{E}$.

Proof. σ-ring game.

□

Theorem 1.12. Let $\mathcal{P}$ be a semiring. $\mathcal{R}(\mathcal{P})$ is precisely the class of all sets of the form $\cup_{i=1}^{n} E_i$ where $E_1, \ldots, E_n$ are disjoint sets of $\mathcal{P}$.

Remark. i.e: For any semiring $\mathcal{P}$, the set of all finite disjoint unions of members of $\mathcal{P}$ is a ring.

Proof. (**Dudley, Proposition 3.2.3**)

Let $\mathcal{L}$ denote the class of all sets of the form $\cup_{i=1}^{n} E_i$ where $E_1, \ldots, E_n$ are disjoint sets of $\mathcal{P}$.

It is trivial to show that $\mathcal{L} \subset \mathcal{R}(\mathcal{P})$. To show that $\mathcal{L} \supset \mathcal{R}(\mathcal{P})$, note that $\mathcal{L} \supset \mathcal{P}$ and use the σ-ring game.

Hint: to show closure under proper differences, let $E = \cup_1^n E_i \in \mathcal{L}$ and $F = \cup_1^m F_j \in \mathcal{L}$ and note

$$E - F = \cup_{i=1}^{n} (E_i - \cup_{j=1}^{m} F_j) = \cup_{i=1}^{n} \cap_{j=1}^{m} (E_i - F_j)$$

□

Corollary 1.13. A finite union of sets of $\mathcal{P}$ may be written as a finite disjoint union of sets in $\mathcal{P}$

Remark. Note: this allows us to omit the word "disjoint" in the statement of the previous theorem.

Proof. Note $E_i \in \mathcal{P}$ implies that $E_i \in \mathcal{R}(\mathcal{P})$. Apply the previous theorem.

□

Theorem 1.14. (Two important properties of generated rings/fields)

Let $\mathcal{E}$ and $\mathcal{F}$ be classes of sets with $\mathcal{E} \subset \mathcal{F}$. Then

1. $\mathcal{S}(\mathcal{E}) \subset \mathcal{S}(\mathcal{F})$ and $\sigma(\mathcal{E}) \subset \sigma(\mathcal{F})$
2. $\mathcal{S}(\mathcal{R}(\mathcal{E})) = \mathcal{S}(\mathcal{E})$

Proof. For (1), note $\mathcal{E} \subset \mathcal{S}(\mathcal{F})$ and apply the definition of generated σ-ring.

For (2), to show $\mathcal{S}(\mathcal{R}(\mathcal{E})) \supset \mathcal{S}(\mathcal{E})$, use (1). To show the opposite inclusion, note that $\mathcal{S}(\mathcal{E})$ is also a ring containing $\mathcal{E}$, and apply the definition of generated σ-ring.

□

1.4 Monotone class theorem, π-λ theorems

Theorem 1.15. (Equivalent definitions of λ-class)

The following are equivalent definitions for λ-class:

1. a nonempty class *containing the whole space* that is closed under proper differences and countable increasing union.
2. a nonempty class *containing the whole space* that is closed under complements and countable disjoint union

   Proof. Note that if $B \subset A$, then $A - B = (X \cup B)^c$.

   □

Theorem 1.16. ($\mathcal{D}$-class theorem)

If $\mathcal{E}$ is a π-class, then $\mathcal{D}(\mathcal{E}) = \mathcal{S}(\mathcal{E})$.

Proof. (**See proof of $\pi - \lambda$ theorem below for easier method**)

If we can show that $\mathcal{D}(\mathcal{E})$ is closed under intersections, then $\mathcal{S}(\mathcal{E}) \supset \mathcal{D}(\mathcal{E})$ and $\mathcal{S}(\mathcal{E}) \subset \mathcal{D}(\mathcal{E})$ are easily checked.

To show that $\mathcal{D}(\mathcal{E})$ is closed under intersections, for some fixed $E \subset X$ define:

$$\mathcal{D}_E = \{F \in \mathcal{D}(\mathcal{E}) : F \cap E \in \mathcal{D}(\mathcal{E})\}$$

Show that $\mathcal{D}_E$ is a $\mathcal{D}$-class. Then it follows that:

1. If $E \in \mathcal{E}$, then $\mathcal{D}_E \supset \mathcal{E}$, so $\mathcal{D}_E \supset \mathcal{D}(\mathcal{E})$.
2. Thus if $E \in \mathcal{E}$ and $G \in \mathcal{D}(\mathcal{E})$, then $G \in \mathcal{D}_E$ so $E \in \mathcal{D}_G$.
3. Thus $\mathcal{D}_G \supset \mathcal{E}$ so $\mathcal{D}_G \supset \mathcal{D}(\mathcal{E})$.

□

Theorem 1.17. (π-λ theorem)

If $\mathcal{E}$ is a π-class, then $\lambda(\mathcal{E}) = \sigma(\mathcal{E})$.

Proof. (**Williams, A1.2, A1.3**)

If we can show that $\lambda(\mathcal{E})$ is closed under intersections, then $\sigma(\mathcal{E}) \supset \lambda(\mathcal{E})$ and $\sigma(\mathcal{E}) \subset \lambda(\mathcal{E})$ are easily checked.

To show that $\lambda(\mathcal{E})$ is closed under intersections, proceed in two steps:

1. Define $D_1 = \{E \in \lambda(\mathcal{E}) : E \cap F \in \lambda(\mathcal{E}),\ \forall F \in \mathcal{E}\}$.

   Show that D_1 is a λ-class which contains $\mathcal{E}$, and that $D_1 = \lambda(\mathcal{E})$.
2. Define $D_2 = \{E \in \lambda(\mathcal{E}) : E \cap G \in \lambda(\mathcal{E}),\ \forall G \in \lambda(\mathcal{E})\}$.

   Use Step 1 to show that D_2 contains $\mathcal{E}$. Then show that it is a λ-class also to conclude.

□

Theorem 1.18. If $\mathcal{E}$ is a field, then $\sigma(\mathcal{E}) = \mathcal{M}(\mathcal{E})$.

Remark. If $\mathcal{M}_0$ is a monotone class containing a field $\mathcal{E}$, then $\mathcal{M}_0 \supset \sigma(\mathcal{E})$.

Proof. (**Bartle, Lemma 10.7**)

Obviously, $\sigma(\mathcal{E}) \supset \mathcal{M}(\mathcal{E})$. To show the reverse inclusion, we can just show that $\mathcal{M}(\mathcal{E})$ is a σ-field containing $\mathcal{E}$.

To show that $\mathcal{M}(\mathcal{E})$ is a σ-field, we show that is a field and then apply closure under monotone limits. To show that $\mathcal{M}(\mathcal{E})$ is a field, we note that $X \in \mathcal{M}(\mathcal{E})$ so it is sufficient to show that it is closed under intersections and relative complements.

Fix some set $E \subset X$ and define:

$$\mathcal{M}_E = \{E \in \mathcal{M}(\mathcal{E}) : E - F, F - E, F \cap E \in \mathcal{M}(\mathcal{E})\}$$

Follow the argument of the above proofs.

□

2 Measures

2.1 Properties of measures

Theorem 2.1. A non-negative and finitely additive set function μ on a semiring $\mathcal{P}$ is monotone and subtractive.

Proof. To show that μ is monotone, use the definition of a semiring. □

Theorem 2.2. (Union bounds-ish) Let μ be a measure on a ring $\mathcal{R}$ and let $E \in \mathcal{R}$.

1. If $\{E_i\} \in \mathcal{R}$ and $\cup_1^\infty E_i \supset E$, then $\mu(E) \leq \sum_1^\infty \mu(E_i)$.
2. If $\{E_i\} \in \mathcal{R}$ is *disjoint* and $\cup_1^\infty E_i \subset E$, then $\mu(E) \geq \sum_1^\infty \mu(E_i)$.

Remark. In (1), it is not assumed that $\cup_1^\infty E_i \in \mathcal{R}$.

Proof. For (1), by a previous lemma we can write $E = \cup_{i=1}^\infty (E \cap E_i) = \cup_{i=1}^\infty G_i$ where the G_i's are **disjoint** and $G_i \subset (E \cap E_i)\ \forall i$.

For (2), establish the result for any n and then send $n \to \infty$. □

Theorem 2.3. (Limits of monotone sequences)

Let μ be a measure on a ring $\mathcal{R}$ and let $\{E_n\}$ be a sequence in $\mathcal{R}$ with $\lim E_n \in \mathcal{R}$ also. If

1. $\{E_n\}$ is increasing and $\lim E_n \in \mathcal{R}$, or
2. $\{E_n\}$ is decreasing, $\mu(E_N) < \infty$ for some N, and $\lim E_n \in \mathcal{R}$.

then $\mu(\lim E_n) = \lim \mu(E_n)$.

Proof. For (1), note $\cup_{i=1}^\infty E_i = \cup_{i=1}^\infty (E_i - E_{i-1})$.

For (2), Form the increasing sequence $(E_N - E_n)$. Note that its limit $\in \mathcal{R}$ and that $\mu(E_N)$ implies $\mu(\lim E_n) < \infty$ and $\mu(\lim(E_N - E_n)) < \infty$. Apply (1). □

Theorem 2.4. (Continuity criterion for measures)

Let μ be a finite, non-negative, additive set function on a ring $\mathcal{R}$. If

1. μ is continuous from below at every $E \in \mathcal{R}$, or
2. μ is continuous from above at $\emptyset$

then μ is a measure on $\mathcal{R}$.

Proof. Let $\{E_n\}$ be a disjoint sequence in $\mathcal{R}$ and denote $E = \cup_{i=1}^\infty E_i \in \mathcal{R}$. Also write $F_n = \cup_{i=1}^n E_i$ and $G_n = E - F_n$.

For (1), note that $\{F_n\}$ is increasing.

For (2), note that $\{G_n\}$ is decreasing with $\lim G_n = \emptyset$, and that $\mu(G_n) = \mu(E) - \mu(F_n)$ since $\mu < \infty$.

□

Lemma 2.5. (Identification lemma)

Suppose μ_1, μ_2 are measures on a σ-ring $\mathcal{S}$ with $\mu_1(A) = \mu_2(A)$ for all $A \in \mathcal{A}$.

If $\mathcal{A}$ is a π-class containing $\emptyset$ with $\mathcal{S}(\mathcal{A}) = \mathcal{S}$ and μ_1 and μ_2 are σ-finite on $\mathcal{A}$, then:

1. μ_1 and μ_2 are σ-finite on $\mathcal{S}$ and
2. $\mu_1(E) = \mu_2(E)$ for all $E \in \mathcal{S}$.

Remark. If $\mathcal{S}$ is a σ-field rather than just a σ-ring, then we must also have $X \in \mathcal{A}$ for the below argument to be valid.

Proof. (For finite measures: **Williams, A1.4**)

To show that μ_1 is actually σ-finite on all of $\mathcal{S}$, show something **stronger**: that all sets which can be covered by a countable union of sets in $\mathcal{A}$ forms a σ-ring (or a σ-field if $X \in \mathcal{A}$) containing $\mathcal{A}$.

Fix $A \in \mathcal{S}$ with $\mu(A) < \infty$. Define:

$$\mathcal{D} = \{E \in \mathcal{S} : \mu_1(A \cap E) = \mu_2(A \cap E)\}$$

Apply the $\mathcal{D}$-class theorem or $\pi - \lambda$ theorem, for fields to show that $\mathcal{D} \supset \mathcal{S}$.

Consider general $E \in \mathcal{S}$. By our first argument, there exists $\{E_n\} \in \mathcal{A}$ such that $\mu_1(E_n) < \infty$ for all n and $E \subset \cup_1^\infty E_n$.

Now write $E = \cup_1^\infty (E \cap E_n)$. By a previous lemma, we can write $E = \cup_1^\infty G_n$ where G_n are disjoint sets in $\mathcal{S}$ with $G_n \subset E \cap E_n$. Express $G_n = E_n \cap (E \cap G_n)$, apply the result from the $\pi - \lambda$ theorem above, and then use countable additivity to conclude.

□

2.2 Outer measure

Theorem 2.6. Let μ^* be an outer measure. To test whether a set E is μ^*-measurable, one only needs to show

$$\mu^*(A) \geq \mu^*(A \cap E) + \mu^*(A \cap E^c) \text{ for every } A \subset X$$

Proof. The other inclusion is obvious.

□

Lemma 2.7. (Properties of μ^*-measurable sets)

Let μ^* be an outer measure and let $\mathcal{S}^*$ be the class of all μ^*-measurable sets. Then for any $E, F \in \mathcal{S}^*$ and $A \subset X$,

1. $\mu^*(A) = \mu^*(A \cap E \cap F) + \mu^*(A \cap E \cap F^c) + \mu^*(A \cap E^c \cap F) + \mu^*(A \cap E^c \cap F^c)$
2. $\mu^*(A \cap (E \cup F)) = \mu^*(A \cap E \cap F) + \mu^*(A \cap E^c \cap F) + \mu^*(A \cap E \cap F^c)$
3. E, F disjoint $\Rightarrow$ $\mu^*(A \cap (E \cup F)) = \mu^*(A \cap E) + \mu^*(A \cap F)$

   Proof. (2) and (3) follow immediately from (1), so we only prove (1):

   Since E is μ^*-measurable, then $\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^c)$. But F is also μ^*-measurable, so:

$$\mu^*(A \cap E) = \mu^*((A \cap E) \cap F) + \mu^*((A \cap E) \cap F^c)$$
$$\mu^*(A \cap E^c) = \mu^*((A \cap E^c) \cap F) + \mu^*((A \cap E^c) \cap F^c)$$

□

Theorem 2.8. ($\mathcal{S}^*$ is a σ-field)

Let μ^* be an outer measure and let $\mathcal{S}^*$ be the class of all μ^*-measurable sets. Then:

1. $\mathcal{S}^*$ is a σ-field.
2. If $\{E_n\}_{n \geq 1}$ is a disjoint sequence in $\mathcal{S}^*$ and $E = \cup_1^\infty E_n$, then

$$\mu^*(E) = \sum_1^\infty \mu^*(E_n)$$

   Remark. (2) in layperson's terms: "The restriction of μ^* to $\mathcal{S}^*$ is a measure on $\mathcal{S}^*$."

   Proof. (**Bartle, Theorem 9.7**)

   The proof has three steps:

   1. Show $\mathcal{S}^*$ is a field:

      Showing closure under complementation is trivial. To show closure under union, use properties (1) and (2) from the Lemma.

   2. Show $\mathcal{S}^*$ is a σ-field:

      $\mathcal{S}^*$ is a field, so we only need to show that if $\{E_n\}$ is a countable *disjoint* sequence in $\mathcal{S}^*$, then $E = \cup_{i=1}^\infty E_i \in \mathcal{S}^*$. Write $F_n = \cup_{i=1}^n E_i$ ($\in \mathcal{S}^*$) and recall that to show that $E \in \mathcal{S}^*$, we need only show:

$$\begin{aligned} \mu^*(A) &\geq \sum_{i=1}^\infty \mu^*(A \cap E_i) + \mu^*)A \cap E^c) \\ &\geq \mu^*(A \cap E) + \mu^*(A \cap E^c) \end{aligned}$$

3. Show that μ^* is a measure on $\mathcal{S}^*$:

   Let $\{E_n\}$ be a countable disjoint sequence in $\mathcal{S}^*$ with $E = \cup_{i=1}^{\infty} E_i$ ($\in \mathcal{S}^*$ as just shown). Using the above inequalities and the fact that μ^* is actually an outer measure, show that:

   $$\mu^*(A) = \sum_{i=1}^{\infty} \mu^*(A \cap E_i) + \mu^*(A \cap E^c)$$

   and set $A = E$.

□

2.3 Extension theorem

Theorem 2.9. (From semiring to ring)

Let μ be a non-negative, finitely additive set function on a semiring $\mathcal{P}$ with $\mu(\emptyset) = 0$. Then:

1. There exists a *unique* non-negative, finitely additive extension ν of μ to $\mathcal{R}(\mathcal{P})$.
2. If μ is countably additive on $\mathcal{P}$ (i.e. is a measure on $\mathcal{P}$), then ν is countably additive on $\mathcal{R}(\mathcal{P})$ (i.e. is a measure on $\mathcal{R}$)
3. If μ is finite (σ-finite) on $\mathcal{P}$, then ν is finite (σ-finite) on $\mathcal{R}(\mathcal{P})$.

   Proof. We prove the parts in order:

   1. Let $E \in \mathcal{R}(\mathcal{P})$. Then $E = \cup_{i=1}^{n} E_i$ where E_i are disjoint sets of $\mathcal{P}$. Define:

      $$\nu(E) = \sum_{i=1}^{n} \mu(E_i)$$

      and check that it is a measure. To check uniqueness, let ν^* be another extension...

   2. Let $E \in \mathcal{R}(\mathcal{P})$ such that $E = \cup_{k=1}^{\infty} E_k$ where the $E_k \in \mathcal{R}(\mathcal{P})$ are disjoint. If $E \in \mathcal{P}$, then write E as a double union and rearrange the non-negative double series to obtain:

      $$\mu(E) = \sum_{k=1}^{\infty} \nu(E_k)$$

      If $E \notin \mathcal{P}$, then $E \in \cup_{j=1}^{n} F_j$ where the $F_j \in \mathcal{P}$ and are disjoint. Then we can write $F_j = \cup_{i=1}^{\infty} (E_i \cap F_j)$. Apply the above result and follow the same steps.

   3. Again, if $E \in \mathcal{R}(\mathcal{P})$, then $E = \cup_{j=1}^{n} F_j$ for $F_j \in \mathcal{P}$. Apply σ-finiteness of μ.

□

Theorem 2.10. (Construction of an outer measure)

Let $\mathcal{R}$ be a ring and let μ be a measure on $\mathcal{R}$. Define the function μ^* by:

$$\mu^*(E) = \inf\left\{\sum_{n=1}^{\infty}\mu(E_n) : \cup_{n=1}^{\infty}E_n \supset E,\ E_n \in \mathcal{R}\ \forall n\right\}$$

for any set E which can be so covered. Otherwise, let $\mu^*(E) = +\infty$. Then μ^* is an outer measure.

Proof. (**Halmos, Theorem 10.A**)

To show that $\mu^*(E) = \mu(E)$ for $E \in \mathcal{R}$, use a previously-proved property of measures. Showing non-negativity and monotonicity for any subset(s) of X is trivial.

To show countable subadditivity, let $\{E_i\}_{i\geq 1}$ be a sequence of sets (**not** necessarily in $\mathcal{R}$). Assume that $\mu(E_i) < \infty$ for all i (otherwise the result is trivial). We want to show:

$$\mu^*(\cup_{i=1}^{\infty}E_i) \leq \sum_{i=1}^{\infty}\mu^*(E_i)$$

Since μ^* is defined as an infimum, for each of the E_i there exists some other sequence $\{E_{ij}\}_{j\geq 1}$ in $\mathcal{R}$ which covers E_i and satisfies:

$$\mu^*(E_i) + \epsilon/2^i \geq \sum_{j=1}^{\infty}\mu(E_{ij})$$

Now cover E with a double union, apply the definition of μ^* and rearrange the double sum. Note that ϵ is arbitrary to obtain the result.

□

Lemma 2.11. With the above notation, $\mathcal{S}(\mathcal{R}) \subset \mathcal{S}^*$.

Proof. It is sufficient to show $\mathcal{R} \supset \mathcal{S}^*$, i.e. that a set $E \in \mathcal{R}$ splits any set additively.

So let $A \subset X$. If $\mu^*(A) = \infty$, then the result is trivial so assume $\mu^*(A) < \infty$. As before, we only need to show:

$$\mu^*(A) \geq \mu^*(A \cap E) + \mu^*(A \cap E^c)$$

Fix $\epsilon > 0$. Then by definition of outer measure, there exists a sequence $\{E_i\}_{i\geq 1}$ in $\mathcal{R}$ which covers A and satisfies:

$$\mu^*(A) + \epsilon \geq \sum_{i=1}^{\infty}\mu(E_i)$$

Now note that since $E \in \mathcal{R}$, the last condition can be written:

$$\mu^*(A) + \epsilon \geq \sum_{i=1}^{\infty}\mu(E_i \cap E) + \sum_{i=1}^{\infty}\mu(E_i \cap E^c)$$

□

Theorem 2.12. (From ring to σ-ring)

Let $\mathcal{R}$ be a ring and let μ be a measure on $\mathcal{R}$. Then

1. There exists a measure $\bar{\mu}$ extending μ to $\mathcal{S}(\mathcal{R})$.
2. If μ is σ-finite, then the extension is unique.

   Proof. Note that the restriction of the outer measure μ^* (defined above) to $\mathcal{S}^*$ is a measure. Note $\mathcal{S}(\mathcal{R}) \subset \mathcal{S}^*$. To show uniqueness under σ-finite μ, apply the identification lemma.

   □

2.4 Completion and approximation

Theorem 2.13. (Completion theorem)

Let μ be a measure on a σ-ring $\mathcal{S}$. Let $\bar{\mathcal{S}}$ be the class of all sets of the form $E \cup N$ where $E \in \mathcal{S}$ and $N \subset A$ where $A \in \mathcal{S}$ such that $\mu(A) = 0$. Then:

1. $\bar{\mathcal{S}}$ is a σ-ring.
2. The set function $\bar{\mu}$ *defined on* $\bar{\mathcal{S}}$ by

$$\bar{\mu}(E \cup N) = \mu(E) \text{ for } E \in \mathcal{S},\ N \subset A \in \mathcal{S},\ \mu(A) = 0$$

   is a complete measure on $\bar{\mathcal{S}}$ and extends μ on $\mathcal{S}$.

   Proof. The proof has three parts:

   1. $\bar{\mu}$ is well-defined:

      Consider two sets in $\bar{\mathcal{S}}$: $E_1 \cup N_1$ and $E_2 \cup N_2$ with the notation defined above. Show $\mu(E_1) = \mu(E_2)$ by noting that $E_1 - E_2 \subset N_2$.

   2. $\bar{\mathcal{S}}$ is a σ-ring:

      To show closure under countable union, use a union bound on the sets of measure zero. To show closure under differences, consider two sets in $\bar{\mathcal{S}}$: $E_1 \cup N_1$ and $E_2 \cup N_2$. Show that:

$$\begin{aligned}(E_1 \cup N_1) - (E_2 \cup N_2) &= \\ &= (E_1 \cap E_2^c \cap A_2^c) \cup (E_1 \cap E_2^c \cap N_2^c \cap A_2) \cup (N_1 \cap E_2^c \cap N_2^c)\end{aligned}$$

      And observe that the last two terms in the RHS union up to a subset of $A_1 \cup A_2$.

   3. $\bar{\mu}$ is a complete measure on $\bar{\mathcal{S}}$:

      To show that $\bar{\mu}$ is a measure, we show countably additivity. Let $\{E_i \cup N_i\}_{i \geq 1}$ be a sequence in $\bar{\mathcal{S}}$ with notation as usual. Note that:

$$\bar{\mu}\Big(\cup_{i=1}^{\infty} (E_i \cup N_i)\Big) = \bar{\mu}\Big((\cup_{i=1}^{\infty} E_i) \cup (\cup_{i=1}^{\infty} N_i)\Big) = \mu(\cup_{i=1}^{\infty} E_i)$$

Then complete the details.

□

Theorem 2.14. (Approximation theorem)

Let $\mathcal{R}$ be a ring and μ a measure on $\mathcal{S}(\mathcal{R})$ which is σ-finite on $\mathcal{R}$.

Then for any $\epsilon > 0$ and $E \in \mathcal{S}(\mathcal{R})$ with $\mu(E) < \infty$, there exists $F \in \mathcal{R}$ such that $\mu(E \Delta F) < \epsilon$.

Proof. It is sufficient to show that $\mu(E-F) \leq \epsilon/2$ and $\mu(F-E) \leq \epsilon/2$.

Since μ is σ-finite on $\mathcal{R}$, then its outer measure extension μ^* is unique. Therefore, $\mu = \mu^*$ on all of $\mathcal{S}(\mathcal{E})$. Then by definition of μ^* there exists a sequence $\{E_n\} \in \mathbb{R}$ such that $\cup^\infty E_n \supset E$ and:

$$\sum_{i=1}^{\infty} \mu(E_i) - \mu(E) \leq \epsilon/2$$

1. $\mu(E - F) \leq \epsilon/2$

   By monotonicity $\lim \mu(\cup_{i=1}^n E_i) = \mu(\cup_{i=1}^\infty E_i)$. Then there exists n_0 such that $F = \cup_{i=1}^{n_0} E_i$ satisfies $\mu(\cup_{i=1}^\infty E_i) - \mu(F) \leq \epsilon/2$.

   Conclude using the fact that $\cup^\infty E_n \supset E$.

2. $\mu(F - E) \leq \epsilon/2$

   Note that $F - E \subset \cup^\infty E_n - E$. Then $\mu(F - E) \leq \mu(\cup^\infty E_n) - \mu(E)$. Apply the obvious union bound and use the fact above.

□

2.5 Lebesgue measure

For the following three theorems, let $\mathcal{P}$ be the semiring of bounded half-open intervals of the form $(a, b]$ and define $\mu((a, b]) = b - a$.

Clearly μ is non-negative with $\mu(\emptyset) = 0$. We show that it is countably additive also:

Lemma 2.15. Let $E_0 \in \mathcal{P}$ and $\{E_n\}_{n \geq 1} \in \mathcal{P}$ be a sequence of disjoint intervals in $\mathcal{P}$ such that $E_n \subset E_0$ for all n. Then

$$\sum_{i=1}^{\infty} \mu(E_i) \leq \mu(E_0)$$

Lemma 2.16. If a bounded closed interval $[a_0, b_0]$ is contained in the union of a finite number of open intervals (a_i, b_i) (for $i = 1, \ldots n$), then

$$b_0 - a_0 \leq \sum_{i=1}^{n} (b_i - a_i)$$

Lemma 2.17. If $E_0, E_1, \ldots$ are sets in $\mathcal{P}$ such that $E_0 \subset \cup_{i=1}^{\infty} E_i$, then

$$\mu(E_0) \leq \sum_{i=1}^{\infty} \mu(E_i)$$

Theorem 2.18. (Existence of Lebesgue measure)

There exists a unique σ-finite measure μ on the σ-field $B(\mathbb{R})$ of Borel sets such that $\mu((a, b]) = b - a$ for all $a < b$.

Theorem 2.19. (Lebesgue-measurability)

Let T be the transformation $Tx = \alpha x + \beta$ $(\alpha \neq 0)$. Then TE is Lebesgue-measurable if and only if E is Lebesgue-measurable.

3 Measurable transformations/functions

Lemma 3.1.

$$\begin{aligned} B(\mathbb{R})^* &= \mathcal{S}\Big\{\{\infty\}, \{-\infty\}, (-\infty, a], -\infty < a < \infty\Big\} \\ &= \mathcal{S}\Big\{\mathbb{R}^*, [-\infty, a], -\infty < a < \infty\Big\} \end{aligned}$$

3.1 Transformations and functions

Theorem 3.2. (Properties of inverse images of transformations)

Let T be a transformation from $D \subset X$ into Y.
Let $G, H, \{G_i\}_{i \geq 1}$ be subsets of Y. Then

1. $T^{-1}(G - H) = T^{-1}(G) - T^{-1}(H)$
2. $T^{-1}(\cup_1^\infty G_i) = \cup_1^\infty T^{-1}(G_i)$
3. $T^{-1}(\cap_1^\infty G_i) = \cap_1^\infty T^{-1}(G_i)$
4. $T^{-1}(G^c) = D - T^{-1}(G)$

Proof. Trivial.

□

Theorem 3.3. (Transformations preserve structure)

Let T be a transformation from $D \subset X$ into Y.
Let $\mathcal{S}$ be a σ-ring on X and $\mathcal{T}$ a σ-ring on Y. Then:

1. $T^{-1}\mathcal{T}$ is a σ-ring on X.
2. $\mathcal{W} = \{G \subset Y : T^{-1}(G) \in \mathcal{S}\}$ is a σ-ring on Y.

For σ-fields, if $D = X$, then (1) holds. If D is measurable, then (2) holds.

Proof. (1) and (2) follow immediately from the previous theorem. To show the extension to σ-fields, note that:

1. If $D = X$ and $\mathcal{T}$ is a σ-field, then $Y \in \mathcal{T}$ and $T^{-1}(Y) = X$.
2. If D is measurable and $\mathcal{S}$ is a σ-field, then $T^{-1}(Y) = D \in \mathcal{S}$.

□

Corollary 3.4. (Transformations preserve generated structure)

Let T be a transformation from $D \subset X$ into Y.
If $\mathcal{G}$ is a class of subsets of Y, then $\mathcal{S}(T^{-1}\mathcal{G}) = T^{-1}\mathcal{S}(\mathcal{G})$.

Proof. For one inclusion, note $T^{-1}\mathcal{G} \subset T^{-1}\mathcal{S}(\mathcal{G})$ and that $T^{-1}(\mathcal{S}(\mathcal{G})$ is a σ-ring.

For the opposite inclusion, note that $\{G \subset Y : T^{-1}(G) \in \mathcal{S}(T^{-1}\mathcal{G})\}$ is a σ-ring containing $\mathcal{G}$ so it is a σ-ring containing $\mathcal{S}(\mathcal{G})$. Thus, the set of subsets of Y that maps back to $\mathcal{S}(T^{-1}\mathcal{G})$ contains $\mathcal{S}(\mathcal{G})$.

□

Theorem 3.5. (Measurability through generating class)

Let $(X, \mathcal{S})$ and $(Y, \mathcal{T})$ be measurable spaces. Let T be a transformation from $D \subset X$ into Y. Let $\mathcal{G}$ be a class of subsets of Y such that $\mathcal{S}(\mathcal{G}) = \mathcal{T}$.

Then T is $\mathcal{S}|\mathcal{T}$-measurable if and only if $T^{-1}\mathcal{G} \subset \mathcal{S}$ (i.e. $T^{-1}(G) \in \mathcal{S}$ for all $G \in \mathcal{G}$).

Proof. The "only if" direction is trivial since $\mathcal{G} \subset \mathcal{S}(\mathcal{G})$.

To show the opposite direction, note that:

$$T^{-1}(\mathcal{G}) \subset \mathcal{S} \Rightarrow \mathcal{S}(T^{-1}\mathcal{G}) \subset \mathcal{S}$$

Apply the previous corollary.

□

Corollary 3.6. With the above notation, $\sigma(T^{-1}\mathcal{T}) = \sigma(T^{-1}\mathcal{G})$.

Proof. First, note $\mathcal{S}(\mathcal{G}) = \mathcal{T}$, so $T^{-1}\mathcal{T} \supset T^{-1}\mathcal{G}$.

Second, note that $T^{-1}\mathcal{G} \subset \sigma(T^{-1}\mathcal{G})$, so T is $\sigma(T^{-1}\mathcal{G})|\mathcal{T}$-measurable by the theorem.

□

3.2 Compositions

Theorem 3.7. (Composition: transformation with transformation)

Let $(X, \mathcal{S})$, $(Y, \mathcal{T})$, and $(Z, \mathcal{W})$ be measurable spaces.
Let T_1 be an $\mathcal{S}|\mathcal{T}$-measurable transformation from $D_1 \subset X$ into Y.
Let T_2 be an $\mathcal{Y}|\mathcal{W}$-measurable transformation from $D_2 \subset Y$ into Z.

Then the composition $T_2 \circ T_1$ defined from $\{x \in D_1 : T_1 x \in D_2\}$ into Z is $\mathcal{S}|\mathcal{W}$-measurable.

Proof. Just check that for any $G \subset Z$,

$$(T_2 T_1)^{-1}(G) = T_1^{-1}(T_2^{-1}(G))$$

□

Theorem 3.8. (Composition: function with transformation)

Let $(X, \mathcal{S})$ and $(Y, \mathcal{T})$ be measurable spaces.
Let T be an $\mathcal{S}|\mathcal{T}$-measurable transformation from $D_1 \subset X$ into Y.
Let g be a $\mathcal{T}$-measurable function from $D_2 \subset Y$ (into $\mathbb{R}^*$).

Then the composition $g \circ T$ is $\mathcal{S}$-measurable.

Proof. Special case of the previous theorem.

□

Theorem 3.9. (Composition: function with function)

Let $(X, \mathcal{S})$ be a measurable space.
Let f be an $\mathcal{S}$-measurable function.
Let g be a $B(\mathbb{R})^*$-measurable function defined on a subset of $\mathbb{R}$.

Then the composition $g \circ f$ is $\mathcal{S}$-measurable.

Proof. See above. □

Theorem 3.10. (Composition: partial converse)

Let X be a space and $(Y, \mathcal{T})$ a measurable space.
Let T be a transformation from (all of) X into Y.

Then a function f defined on X is $T^{-1}\mathcal{T}$-measurable if and only if there exists a $\mathcal{T}$-measurable function g defined on Y such that $f = g \circ T$.

Proof. By simple functions. See (Leadbetter, Theorem 3.5.3).

3.3 Combining measurable functions

Lemma 3.11. Let $f_1, \ldots, f_n$ be measurable functions defined (respectively) on $D_1, \ldots, D_n \subset X$. Let h be defined on $H = H_1 \cup H_2 \cup \ldots \cup H_n$ where the H_i are disjoint measurable sets with $H_i \subset D_i$, by:

$$h(x) = f_i(x) \text{ for } x \in H_i$$

Then h is measurable.

Proof.

Corollary 3.12. Let f be a measurable functions defined on $D \subset X$ and let h be its restriction to a measurable subset $H \subset D$. Then h is measurable.

Proof.

Theorem 3.13. (Linear combinations)

Let f, g be measurable functions. Then $f + g$ is measurable and $a \cdot f$ is measurable for any $a \in \mathbb{R}$.

Proof.

Theorem 3.14. (Exponentiation)

If $f(x)$ is a measurable function, then

1. $|f(x)|^a$ is measurable for any $a \in \mathbb{R}$
2. $f^n(x)$ is measurable for $n = 1, 2, \ldots$

Proof.

Theorem 3.15. (Multiplication)

If f, g are measurable functions, then $f \cdot g$ and f/g are measurable.

Proof.

Theorem 3.16. (Maximum and minimum)

If f, g are measurable functions, then $\max(f, g)$, $\min(f, g)$, f^+, f^-, and $|f|$ are all measurable.

Proof.

Theorem 3.17. (Limits)

Let $\{f_n\}$ be a sequence of measurable functions. Let D denote the set on which all the f_n's are defined and on which $\{f_n\}$ converges (to a finite or infinite value).

Then D is a measurable set and $f = \lim_{n\to\infty} f_n(x)$ defined on D is a measurable function.

Proof.

3.4 Simple functions

Theorem 3.18. (Two properties of simple functions)

1. Finite linear combinations and products of simple functions are simple functions
2. f is simple if and only if, for every $x \in X$

$$f(x) = \sum_{i=1}^{n} a_i \cdot \mathbb{1}_{E_i}(x)$$

where $E_1, \ldots, E_n$ are disjoint measurable sets that union to X, and $a_1, \ldots, a_n \in \mathbb{R}$.

Proof.

Theorem 3.19. (Approximation by simple functions)

Let f be a *non-negative* measurable function defined on all of X. Then there exists an *increasing* sequence of *non-negative* simple functions $\{f_n\}$ such that $f_n(x) \to f(x)$ for all $x \in X$.

Proof. Let $\{f_n\}$ be the sequence of *staircase functions* defined by:

$$f_n(x) = \begin{cases} \frac{i-1}{2^n}, & \frac{i-1}{2^n} \leq f(x) \leq \frac{i}{2^n} \\ n, & f(x) \geq n \end{cases}$$

Note i runs from $1, \ldots, n2^n$.

Obviously f_n is simple. To show that it is increasing in n (for every x), note that if $f_n(x) = (i-1)/2^n$, then:

$$\frac{2i-2}{2^{n+1}} \leq f(x) \leq \frac{2i}{2^{n+1}}$$

Note that $f(x)$ is either between $(2i-2)/2^{n+1}$ and $(2i-1)/2^{n+1}$, or between $(2i-1)/2^{n+1}$ and $(2i)/2^{n+1}$. Conclude.

To show that $f_n \to f$ for every x, consider two cases: $f(x) < \infty$ and $f(x) = \infty$. Show that, for any $\epsilon > 0$, $\exists N$ such that for all $n > N$, $f(x) - f_n(x) < \epsilon$.

For $f(x) < \infty$, consider $N > f(x)$.

□

Corollary 3.20. Let f be a measurable function defined on all of X. Then there exists an sequence of simple functions $\{f_n\}$ such that $f_n(x) \to f(x)$ for all $x \in X$.

Remark. In fact, $\{f_n\}$ may be taken so that $\{|f_n|\}$ is increasing.

Proof.

Theorem 3.21.

3.5 Measure spaces and induced measures

Theorem 3.22. (Measurability and completeness)

Let $(X, \mathcal{S}, \mu)$ be a measure space and let f, g be defined on subsets of X. If f is measurable, $f = g$ a.e. **and** μ is complete, then g is also measurable.

Proof.

Theorem 3.23. (Induced measure)

Let $(X, \mathcal{S}, \mu)$ be a measure space and $(Y, \mathcal{T})$ be a measure space. Let T be a measurable transformation from $D \subset X$ into Y. Then the set function μT^{-1} defined on $\mathcal{T}$ by

$$(\mu T^{-1})(G) = \mu(T^{-1}(G)), \quad G \in \mathcal{T}$$

is a measure.

Proof.

4 The integral

4.1 Integration of non-negative simple functions

Lemma 4.1. Let f be a non-negative simple function, $f(x) = \sum_{i=1}^{n} a_i \mathbb{1}_{E_i}(x)$, where $E_1, ..., E_n$ are disjoint sets in $\mathcal{S}$ with union X and $a_i \geq 0$. Then the extended non-negative real number

$$\sum_{i=1}^{n} a_i \mu(E_i)$$

does not depend on the particular representation of f.

Proof. Let another representation of f be given by $f(x) = \sum_{j=1}^{m} b_j \mathbb{1}_{F_j}(x)$, where the F_j are disjoint sets in $\mathcal{S}$ whose union is X and $b_j \geq 0$. Show $\sum_{i=1}^{n} a_i \mu(E_i) = \sum_{j=1}^{m} b_j \mu(F_j)$ by noting that

$$X = \sum_{i=1}^{n} (\cup_{j=1}^{m} (E_i \cap F_j) = \sum_{j=1}^{m} (\cup_{i=1}^{n} (E_i \cap F_j)$$

.

Lemma 4.2. (A few results re: simple functions)

1. Any two simple functions f and g may be represented as below, in terms of the **same** E_i satisfying the usual properties

$$f = \sum a_i \mathbb{1}_{E_i}, g = \sum b_i \mathbb{1}_{E_i}$$

2. If f, g are non-negative simple functions and a, b are non-negative real numbers, then

$$\int (af + bg)\,\mathrm{d}\mu = a \int f\,\mathrm{d}\mu + b \int g\,\mathrm{d}\mu$$

3. If f, g, are non-negative simple functions such that $f(x) \geq g(x)$ for all x, then

$$\int f\,\mathrm{d}\mu \geq \int g\,\mathrm{d}\mu$$

Proof. Immediate.

Lemma 4.3. If f_n is an increasing sequence of non-negative simple functions and g is a non-negative simple function such that $\lim_{n\to\infty} f_n(x) \geq g(x)$ for all $x \in X$, then

$$\lim_{n\to\infty} \int f_n\,\mathrm{d}\mu \geq \int g\,\mathrm{d}\mu$$

Proof. Since g is simple, write $g(x) = \sum_{1}^{m} a_i \mathbb{1}_{E_i}$ with the E_i's as usual. Then $\int g\,\mathrm{d}\mu = \sum_{1}^{m} a_i \mu(E_i)$.

Case one: $\int g \, \mathrm{d}\mu = \infty$.

Then for some p $(1 \leq p \leq m)$, $a_p > 0$ and $\mu(E_p) = \infty$. Let ϵ be such that $0 < \epsilon < a_p$ and note that the sequence $A_n = \{x : f_n(x) > g(x) - \epsilon\}$ is monotone non-decreasing. It follows that $\lim_{n\to\infty} A_n \cap E_p = E_p$. Show $\lim_{n\to\infty} \int f_n \, \mathrm{d}\mu = \infty$.

Case two: $\int g \, \mathrm{d}\mu < \infty$.

Let $A = \{x : g(x) > 0\} = \cup\{E_i : a_i > 0\}$. Let a be the minimum non-zero a_i. Note $\mu(A) = \sum_{a_i > 0} \mu(E_i) < \infty$ since $\int g \, \mathrm{d}\mu < \infty \Rightarrow \mu(E_i) < \infty$ for each i with $a_i > 0$. Define A_n as above, with ϵ such that $0 < \epsilon < a$. Note:

$$f_n \geq f_n \mathbb{1}_{A_n \cap A} \geq (g - \epsilon)\mathbb{1}_{A_n \cap A} \geq 0$$

Take integrals and then limits to show the result.

Theorem 4.4. Let f be a non-negative measurable function defined on X and let f_n be an increasing sequence of non-negative simple functions such that $f_n(x) \to f(x)$ for all $x \in X$. Then the extended non-negative real number $\lim_{n\to\infty} \int f_n \, \mathrm{d}\mu$ does not depend on the particular sequence f_n.

Proof. Let g_n be another increasing sequence of non-negative simple functions with $\lim_{n\to\infty} g_n(x) = f(x)$ for all $x \in X$. Note $\lim_{n\to\infty} f_n(x) \geq g_m(x)$ for fixed m. Apply the previous Lemma.

Lemma 4.5. Let f and g be non-negative measurable functions on **all** of X.

1. If $a \geq 0$, $b \geq 0$ then $\int (af + bg) \, \mathrm{d}\mu = a\int f \, \mathrm{d}\mu + b\int g \, \mathrm{d}\mu$
2. If $f(x) \geq g(x)$ for all $x \in X$, then $\int f \, \mathrm{d}\mu \geq \int g \, \mathrm{d}\mu$

Proof. Linearity of the limit operator and a previous result re: integrals of simple functions.

Theorem 4.6. Let f be a non-negative measurable function on **all** of X.

1. If E is a measurable set such that $\mu(E) = 0$, then $\int_E f \, \mathrm{d}\mu = 0$.
2. If g is another measurable function on X such that $f = g$ a.e., then $\int f \, \mathrm{d}\mu = \int g \, \mathrm{d}\mu$.

Proof. For the first result, note that if $f_n = \sum_i a_i \mathbb{1}_{E_i}$, then $f_n \mathbb{1}_E = \sum_i a_i \mathbb{1}_{E_i \cap E}$. For the second, note that $\int f = \int f \mathbb{1}_E + \int f \mathbb{1}_{E^c}$.

Lemma 4.7. Let f be a measurable function defined and non-negative a.e. Then the integral of f is unambiguously defined by $\int f \, \mathrm{d}\mu = \int g \, \mathrm{d}\mu$ where g is any non-negative measurable function on X with $f = g$ a.e.

Proof. Immediate.

Lemma 4.8. Let f and g be measurable functions defined (**not** necessarily on all of X) and non-negative a.e.

1. for $a \geq 0, b \geq 0$, af + bg is also measurable, defined, and non-negative a.e. and $\int (af + bg)\,\mathrm{d}\mu = a\int f\,\mathrm{d}\mu + b\int g\,\mathrm{d}\mu$
2. If $f \geq g$ a.e. then $\int f\,\mathrm{d}\mu \geq \int g\,\mathrm{d}\mu$
3. If $f = g$ a.e. then $\int f\,\mathrm{d}\mu = \int g\,\mathrm{d}\mu$
4. If $E \in \mathcal{S}$ and $\mu(E) = 0$, then $\int_E f\,\mathrm{d}\mu = 0$

Proof. Let f' and g' be functions defined on all of X with $f = f'$ a.e. and $g = g'$ a.e. Then the results are immediate from a previous theorem. □

4.2 Integrability, properties of the integral

Theorem 4.9.

1. If f is integrable, it is finite a.e.
2. If f is measurable, defined a.e., and $E \in \mathcal{S}$ with $\mu(E) = 0$, then $f\mathbb{1}_E$ is integrable and $\int_E f\,\mathrm{d}\mu = \int f\mathbb{1}_E\,\mathrm{d}\mu = 0$

Proof. For the first result, let $E = f^{-1}\{\infty\} = f_+^{-1}\{\infty\}$ and note that $f_+ \geq n\mathbb{1}_E$ for all $n \in \mathbb{N}$. The second follows immediately from a previous Lemma. □

Theorem 4.10. Let f and g be measureable, defined a.e., and such that f = g a.e. on $(X, \mathcal{S}, \mu)$. Then:

1. If f is integrable, then so is g and $\int g = \int f$
2. If f is not integrable but $\int f$ is defined, then g is not integrable but $\int g$ is defined and $\int f = \int g$ (i.e. $\pm\infty$)
3. If $\int f$ is not defined, then $\int g$ is not defined
4. If f is an integrable function, there exists a finite-valued integrable function h defined on X with h = f a.e. (and hence $\int h = \int f$)

Proof. Split $f = f_+ - f_-$ and use the properties of integrals of non-negative (measurable) functions. For the last result, note that f integrable $\Rightarrow f$ finite a.e. □

Theorem 4.11. (Linearity of the integral)

1. If f and g are integrable, then so is $(f + g)$ and $\int (f + g) = \int f + \int g$
2. If f is integrable, then for all $a \in \mathbb{R}$, (af) is integrable and $\int af = a\int f$

Proof. As above. □

Theorem 4.12. Let f and g be measurable functions defined a.e. with f $\geq$ g a.e. and such that $\int f, \int g$ are defined. Then $\int f\,\mathrm{d}\mu \geq \int g\,\mathrm{d}\mu$

Proof. As above. □

Theorem 4.13. Let f be a measurable function defined a.e. Then the following conditions are equivalent:

1. f is integrable.

2. f_+ and f_- are both integrable.
3. $|f|$ is integrable.

Also, f integrable $\Rightarrow |\int f\, \mathrm{d}\mu| \le \int |f|\, \mathrm{d}\mu$.

Proof. Immediate from linearity of the integral and the previous theorem.

Corollary 4.14. Let f be integrable and g be measurable and defined a.e. such that $|g| \le |f|$ a.e. Then g is integrable.

Theorem 4.15. Let f be a measurable function defined and non-negative a.e. with $\int f\, \mathrm{d}\mu = 0$. Then f = 0 a.e.

Proof. Let $E = \{x : f(x) > 0\}$. Show $\mu(E) = 0$ by considering $f\mathbb{1}_{E_n}$ where $E_n = \{x : f(x) \ge \frac{1}{n}\}$, for $n = 1, 2, 3, \ldots$.

Theorem 4.16. If f is integrable and $\int_E f\, \mathrm{d}\mu = 0$ for all $E \in \mathcal{S}$, then f = 0 a.e.

Proof. Define E as above. Consider the possible values of $f\mathbb{1}_E$ a.e. and on E only.

Corollary 4.17. If f and g are integrable and $\int_E f\, \mathrm{d}\mu = \int_E g\, \mathrm{d}\mu$ for all $E \in \mathcal{S}$, then f = g a.e.

Theorem 4.18. If f is integrable, then

1. $\mu\{x : |f(x)| \ge \epsilon\} < \infty$ for every $\epsilon > 0$
2. $N_f = \{x : f(x) \ne 0\}$ has σ-finite measure

Proof. For (1), let $E = \{x : |f(x)| \ge \epsilon\}$ and consider $|f|\mathbb{1}_E$. For (2), express N_f as a union of sets and apply the first result.

Theorem 4.19. If $f : E \to \mathbb{R}$ be measurable. If E has finite measure and f is bounded, then f is integrable.

Proof.

4.3 Convergence of integrals

Lemma 4.20. (Baby monotone convergence theorem)

Let $\{f_n\}$ be an increasing sequence of non-negative measurable functions defined on **all** of X. Let f be a non-negative measurable function defined on all of X such that $f_n(x) \to f(x)$ for all $x \in X$. Then

$$\int f_n\, \mathrm{d}\mu \to \int f\, \mathrm{d}\mu$$

Proof. Since each f_n is measurable, then each f_n has a sequence of non-negative simple functions $\{f_{n,k}\}_{k=1}^{\infty}$ increasing to it (in k).

Argue that the sequence $\{g_k\}$ defined by $g_k(x) = \max_{n \leq k} f_{n,k}(x)$ is also simple and increasing. Establish that, for all x and $n \leq k$,

$$f_{n,k}(x) \leq g_k(x) \leq f_k(x)$$

and take limits in the correct order to show that $\{g_k\}$ converges to f.

Take integrals of the above expression and then take limits again to show the main result (using the definition of an integral as the limit of integrals of simple functions).

Monotone Convergence Theorem. Let $\{f_n\}$ be a sequence of non-negative measurable functions each defined **a.e.** with $f_n(x) \leq f_{n+1}(x)$ a.e. for each n. Let f be a measurable function defined and non-negative a.e. such that $f_n(x) \to f(x)$ a.e. Then

$$\int f_n \, \mathrm{d}\mu \to \int f \, \mathrm{d}\mu$$

Proof. Find $E \in \mathcal{S}$ such that $\mu(E^c) = 0$ and the conditions of the theorem are satisfied. Define f_n', f' such that $f_n'(x) = f_n(x)$ and $f'(x) = f(x)$ for $x \in E$ and $f_n'(x) = f'(x) = 0$ for $x \in E^c$. Apply the previous Lemma and a previous result re: integrals of functions that are equal a.e.

Corollary 4.21. Let $\{f_n\}$ be a sequence of measurable functions defined and non-negative a.e. Then $\sum_1^{\infty} f_n$ is measurable, defined and non-negative a.e. and

$$\int (\sum_{n=1}^{\infty} f_n) \, \mathrm{d}\mu = \sum_{n=1}^{\infty} \int f_n \, \mathrm{d}\mu$$

Theorem 4.22. Let $\epsilon > 0$. If f is integrable, then $\exists \delta > 0$ such that $\int_E |f| \, \mathrm{d}\mu < \epsilon$ whenever $E \in \mathcal{S}$ and $\mu(E) < \delta$. In particular, if $\mu(E_n) \to 0$ as $n \to \infty$, then $\int_{E_n} f \, \mathrm{d}\mu \to 0$ as $n \to \infty$.

Proof. Define f_n by:

$$f_n(x) = \begin{cases} |f|, & \text{if } |f| \leq n \\ n, & \text{otherwise} \end{cases}$$

Note $\lim_{n \to \infty} f_n = |f|$ a.e., so write

$$\int_E |f| \, \mathrm{d}\mu = \int_E |f| - f_N \, \mathrm{d}\mu + \int_E f_N \, \mathrm{d}\mu$$

Apply an obvious bound to the second RHS term and use MCT to make the first RHS term appropriately small.

□

Fatou's Lemma. Let $\{f_n\}$ be a sequence of measurable functions each defined and non-negative a.e. Then

$$\liminf_{n\to\infty} \int f_n \, \mathrm{d}\mu \geq \int (\liminf_{n\to\infty} f_n) \, \mathrm{d}\mu$$

Proof. Define $g_n(x) = \inf_{k\geq n} f_k(x)$. Note that for all $k \geq n$, $g_n \leq f_k$ a.e. and apply monotone convergence.

□

Dominated Convergence Theorem. Let $\{f_n\}$ be integrable functions on $(X, \mathcal{S}, \mu)$ such that $f_n \to f$ a.e. for some measurable function f. Also, let g be an integrable function such that $|f_n| \leq g$ a.e. for each $n = 1, 2, \ldots$. Then f is also integrable and

$$\int |f_n - f| \, \mathrm{d}\mu \to 0 \text{ as } n \to \infty$$

Proof. Integrability of f is obvious. To show convergence of the integral, note:

1. $(2|g| - |f_n - f|)$ is defined and non-negative a.e.
2. $\lim(2|g| - |f_n - f|) = 2|g|$ since $f_n \to f$.

Take integrals and apply Fatou's Lemma.

□

Corollary 4.23. Under the conditions for Dominated Convergence, then also

$$\int f_n \, \mathrm{d}\mu \to \int f \, \mathrm{d}\mu$$

4.4 Real line applications

Definition. Let $\mathcal{B}$ be the Borel sets of $\mathbb{R}$. Let μ_F be a Lebesgue-Stieltjes measure on $\mathcal{B}$ corresponding to a non-decreasing right continuous function F defined on $\mathbb{R}$.

If g is a Borel-measurable function such that $\int g \, \mathrm{d}\mu_F$ is defined, then the **Lebesgue-Stieltjes integral of g with respect to F** is

$$\int_{\mathbb{R}} g \, \mathrm{d}F = \int_{-\infty}^{\infty} g(x) \, \mathrm{d}F(x) = \int_{\mathbb{R}} g \, \mathrm{d}\mu_F$$

For the case of $F(x) = x$, then the **Lebesgue integral of g** is

$$\int_{-\infty}^{\infty} g(x) \, \mathrm{d}x = \int_{\mathbb{R}} g \, \mathrm{d}m$$

where m is the Lebesgue measure on the Borel sets $\mathcal{B}$ or on the Lebesgue-measurable sets $\mathcal{L}$ as appropriate.

Remark. For g above, we have also $\int_{\mathbb{R}} g \, \mathrm{d}F = \int_{\mathbb{R}} g \, \mathrm{d}\overline{\mu}_F$ where $\overline{\mu}_F$ is the completion of μ_F on the corresponding completed Borel σ-field.

Theorem 4.24. Let g be a measurable function. Then $g \in L_1$ if and only if

$$\lim_{n\to\infty} \int_{-n}^{n} |g(x)| \, \mathrm{d}x < \infty$$

Proof. For all measurable g, monotone convergence gives us

$$\int_{-\infty}^{\infty} |g(x)| \, \mathrm{d}x = \lim_{n\to\infty} \int_{-n}^{n} |g(x)| \, \mathrm{d}x$$

Thus $\lim_{n\to\infty} \int_{-n}^{n} |g(x)| \, \mathrm{d}x < \infty \iff |g| \in L_1$, and g is measurable so $|g| \in L_1$ may be replaced by $g \in L_1$.

Remark. If $g \in L_1$ then dominated convergence gives us

$$\int_{-\infty}^{\infty} g(x) \, \mathrm{d}x = \lim_{n\to\infty} \int_{-n}^{n} g(x) \, \mathrm{d}x$$

Thus for some measurable function g we can first check whether it is integrable using the above theorem, and then evaluate the integral using the limit formulation given here.

Theorem 4.25. Let E_0 be a fixed measurable subset of $(X, \mathcal{S}, \mu)$ and let the measure μ_0 be defined on $\mathcal{S}$ by $\mu_0(E) = \mu(E \cap E_0)$ for $E \in \mathcal{S}$. Then

$$\int f \, \mathrm{d}\mu_0 = \int_{E_0} f \, \mathrm{d}\mu$$

for any f for which $\int f \, \mathrm{d}\mu_0$ is defined.

Proof. Note that

$$\begin{aligned} \int f \, \mathrm{d}\mu_0 &= \int_{E_0} f \, \mathrm{d}\mu_0 + \int_{E_0^c} f \, \mathrm{d}\mu_0 \\ &= \int_{E_0} f \, \mathrm{d}\mu_0 \end{aligned}$$

Decompose into simple functions and use the definition of μ_0.

5 Absolute continuity and related topics

Theorem 5.1. Let μ be a signed measure on a σ-ring $\mathcal{S}$.

1. If $E, F \in \mathcal{S}, E \subset F$ and $|\mu(F)| < \infty$, then $|\mu(E)| < \infty$.
2. If $E, F \in \mathcal{S}, E \subset F$ and $|\mu(E)| < \infty$, then $\mu(F \setminus E) = \mu(F) - \mu(E)$.
3. If $\{E_n\}_{n=1}^{\infty}$ is a disjoint sequence of sets in $\mathcal{S}$ such that $|\mu(\cup_{n=1}^{\infty} E_n)| < \infty$ then the series $\sum_{n=1}^{\infty} \mu(E_n)$ converges absolutely.
4. If $\{E_n\}_{n=1}^{\infty}$ is a monotone sequence of sets in $\mathcal{S}$ and if $|\mu(E_n)| < \infty$ for some integer n in the case when $\{E_n\}$ is a decreasng sequence, then
$$\mu(\lim_n E_n) = \lim_n \mu(E_n)$$

Proof. Trivial.

□

5.1 Hahn and Jordan decomposition

Hahn Decomposition. If μ is a signed measure on $(X, \mathcal{S})$ then there exists two disjoint sets A, B such that A is positive, B is negative, and $A \cup B = X$.

Proof. (**Halmos, Theorem 29.A**)

Define $\lambda = \inf\{\mu(E) : E \text{ negative}\}$ and let $\{B_n\}_{n=1}^{\infty}$ be a sequence such that $\lambda = \lim_{n\to\infty} \mu(B_n)$ with $B = \cup_{n=1}^{\infty} B_n$.

1) Show B is negative. Trivial.

2) Show $\mu(B) = \lambda$ (so $-\infty < \lambda \leq 0$) by noting that $B = (B \setminus B_n) \cup B_n$ for each n.

3) Let $A = X \setminus B$. Show $F \subset A$ negative $\Rightarrow F$ is null.

4) Show $A = X \setminus B$. By contradiction.

Jordan Decomposition. Let μ be a signed measure on $(X, \mathcal{S})$. If $X = A \cup B$ is a Hahn decomposition of X for μ, then the set functions μ_+, μ_- defined on $\mathcal{S}$ for each $E \in \mathcal{S}$ by:

$$\mu_+(E) = \mu(E \cap A)$$

$$\mu_-(E) = \mu(E \cap B)$$

are measures on $\mathcal{S}$ and at least one of μ_+, μ_- are finite with $\mu = \mu_+ + \mu_-$.

The measures $\mu_+ + \mu_-$ do not depend on the particular Hahn decomposition.

Proof. (**Halmos, Theorem 29.B**)

The proof that $\mu_+ + \mu_-$ is straightforward. To prove that $\mu_+ + \mu_-$ do not depend on the particular Hahn decomposition chosen, consider two Hahn decompositions of X and show that μ is equal on the two decompositions.

Theorem 5.2. Let $(X, \mathcal{S}, \mu)$ be a measure space and f a measurable function defined a.e. on X. Also assume either $f_+ \in L_1$ or $f_- \in L_1$. Then the set function ν defined for each $E \in \mathcal{S}$ by

$$\nu(E) = \int_E f \, \mathrm{d}\mu$$

is a signed measure on $\mathcal{S}$. If $f \in L_1$ then ν is a finite signed measure.

Proof. To show countable additivity of ν, use monotone convergence.

Lemma 5.3. Let μ, ν be signed measures on the σ-field $\mathcal{S}$ which are equal on a semiring $\mathcal{P}$ with $S(\mathcal{P}) = \mathcal{S}$. If μ (and hence also ν) is σ-finite on $\mathcal{P}$, then $\mu = \nu$ on $\mathcal{S}$.

Proof. Decompose the signed measures and apply the corresponding lemma for measures.

Theorem 5.4. Let μ be a signed measure on $(X, \mathcal{S})$. Then:

1. If $f \in L_1(|\mu|)$, then $|\int f \, \mathrm{d}\mu| \leq \int |f| \, \mathrm{d}|\mu|$.
2. (Dominated Convergence). Let $\{f_n\}$ be a sequence of functions in $L_1(|\mu|)$ and let $g \in L_1(|\mu|)$ such that $|f_n| \leq |g|$ a.e. $(|\mu|)$ for each $n = 1, 2, \ldots$. If f is a measurable function such that $f_n \to f$ a.e. $(|\mu|)$, then $f \in L_1$ and

$$\int |f_n - f| \, \mathrm{d}\mu \to 0, \quad \int f_n \, \mathrm{d}\mu \to \int f \, \mathrm{d}\mu \quad \text{as } n \to \infty$$

Proof. For (i), recall that if $f \in L_1$ and μ is a measure, then $|\int f \, \mathrm{d}\mu| \leq \int |f| \, \mathrm{d}\mu$. For (ii), apply dominated convergence w.r.t. the measure $|\mu|$.

Theorem 5.5. Let $(X, \mathcal{S})$ and $(Y, \mathcal{T})$ be measurable spaces, μ a signed measure on $\mathcal{S}$ and T a measurable transformation defined a.e. $(|\mu|)$ on X into Y. Then the set function μT^{-1} defined on $\mathcal{T}$ by

$$(\mu T^{-1})(E) = \mu(T^{-1} E), \quad E \in \mathcal{T}$$

is a signed measure on $\mathcal{T}$. If f is a $\mathcal{T}$-measurable function defined a.e. (μT^{-1}) on Y such that $fT \in L_1(|\mu|)$, then $f \in L_1(|\mu T^{-1}|)$ and

$$\int_Y f \, \mathrm{d}\mu T^{-1} = \int_X fT \, \mathrm{d}\mu$$

Proof. Complicated. Probably don't need to know it.

□

5.2 Radon-Nikodym theorem

Theorem 5.6. If μ and ν are signed measures on $(X, \mathcal{S})$, then the following are equivalent:

1. $\nu \ll \mu$
2. $\nu_+ \ll \mu$ and $\nu_- \ll \mu$
3. $|\nu| \ll |\mu|$

Proof.

Lemma 5.7. (Baby Radon-Nikodym Theorem)

Let $(X, \mathcal{S}, \mu)$ be a finite measure space and ν a finite measure on $\mathcal{S}$. Then $\exists \nu_1, \nu_2$, uniquely determined finite measures on $\mathcal{S}$ such that

$$\nu = \nu_1 + \nu_2, \quad \nu_1 \ll \mu, \quad \nu_2 \perp \mu$$

and an essentially unique μ-integrable (non-negative) function f such that for all $E \in \mathcal{S}$

$$\nu_1(E) = \int_E f \, \mathrm{d}\mu$$

Proof. We prove each assertion separately:

1. **Uniqueness of ν_1, ν_2, and f**

   Uniqueness of f is trivial. To show uniqueness of the decomposition, let $\nu = \nu_3 + \nu_4$ be another decomposition satisfying the conditions. Show that $\lambda = \nu_1 - \nu_3 = \nu_4 - \nu_2$ and that $\lambda \ll \mu$ **and** $\lambda \perp \mu$.

2. **Existence of f and ν_1**

   We define f in a certain way to make showing the existence of ν_2 possible. Let $\mathcal{K}$ = the class of all non-negative measurable functions g on X such that $\int_E g \, \mathrm{d}\mu \leq \nu(E)$ for all $E \in \mathcal{S}$.

   We want f to be the element of $\mathcal{K}$ that maximizes $\int f \, \mathrm{d}\mu$. Let:

   $$\alpha = \sup \left\{ \int_X f \, \mathrm{d}\mu : f \in \mathcal{K} \right\}$$

   and $\{f_n\}$ be a sequence of functions in $\mathcal{K}$ such that $\int_X f_n \, \mathrm{d}\mu \to \alpha$.

   Now define $g_n(x) = \max\{f_1(x), \ldots, f_n(x)\}$. For any given $E \in \mathcal{S}$, split E into disjoint E_i's by:

   $$E_1 = \{x : g_n(x) = f_1(x)\}$$
   $$E_2 = \{x : g_n(x) = f_2(x)\} \setminus E_1 \ldots$$

   to show that $g_n \in \mathcal{K}$. Then note that g_n is monotone so can define:

   $$f(x) = \lim_{n \to \infty} g_n(x)$$

Use MCT to establish $f \in \mathcal{K}$ and $\int_X f \, \mathrm{d}\mu = \alpha$ (hint: $g_n \geq f_n \ \forall n$). Define ν_1 by $\nu_1(E) = \int_E f \, \mathrm{d}\mu$ and check conditions.

3. **Existence of ν_2**

   Define ν_2 by $\nu_2(E) = \nu(E) - \nu_1(E)$ for all $E \in \mathcal{S}$

   To show $\nu_2 \perp \mu$, define the finite signed measure $\lambda_n = \nu_2 - n^{-1}\mu$. Let $X = A_n \cup B_n$ be a Hahn decomposition of X w.r.t. λ_n. Also, define $h_n = f + n^{-1}\mathbb{1}_{A_n}$.

   Show that $h_n \in \mathcal{K}$, so that $\mu(A_n) = 0$. Then set $A = \cup_{n=1}^{\infty} A_n$. Note $A^c \subset A_n^c$, so $\lambda_n(A^c) \leq 0$. Conclude that $\nu_2(A^c) = 0$, using finiteness of μ.

□

Lebesgue Decomposition Theorem. Let $(X, \mathcal{S}, \mu)$ be a σ-finite measure space and ν a σ-finite signed measure on $\mathcal{S}$. Then $\exists \nu_1, \nu_2$, uniquely determined σ-finite measures on $\mathcal{S}$ such that

$$\nu = \nu_1 + \nu_2, \quad \nu_1 \ll \mu, \quad \nu_2 \perp \mu$$

If ν is a measure then so are ν_1 and ν_2.

Proof. We prove the result for the case when ν and μ are σ-finite measures. To extend that result to the case when ν is a σ-finite signed measure, use Jordan decomposition.

To show 1), note that if ν and μ are σ-finite measures then $X = \cup_{n=1}^{\infty} X_n$, where X_n's are disjoint measurable with finite ν- and μ-measure. Define:

$$\mu^{(n)}(E) = \mu(E \cap X_n) \quad \text{and} \quad \nu^{(n)}(E) = \nu(E \cap X_n)$$

Apply the previous Lemma to obtain the decomposition of $\nu^{(n)}$. Finally define:

$$\nu_1(E) = \sum_n \nu_1^{(n)}(E) \quad \text{and} \quad \nu_2(E) = \sum_n \nu_2^{(n)}(E)$$

And show $\nu_1 \ll \mu, \quad \nu_2 \perp \mu$. Uniqueness is shown similarly.

Radon-Nikodym Theorem. Let $(X, \mathcal{S}, \mu)$ be a σ-finite measure space and ν a σ-finite signed measure on $\mathcal{S}$. If $\nu \ll \mu$ then there is an essentially unique finite-valued measurable function f on X such that, for all $E \in \mathcal{S}$,

$$\nu(E) = \int_E f \, \mathrm{d}\mu$$

f is μ-integrable iff ν is finite. If ν is a measure then f is non-negative.

Proof. As before we show the existence of f for the case of σ-finite measure. The result for σ-finite signed measure follows from applying Jordan decomposition.

Decompose $X = \cup_{n=1}^{\infty}$ and define $\mu^{(n)}, \nu^{(n)}$ as in the proof of Lebesgue decomposition. Using the Radon-Nikodym theorem for finite measures ($\mu^{(n)}$ and $\nu^{(n)}$ are finite measures), express $\nu^{(n)}$ in integral form and obtain a sequence of functions $\{f_n\}$.

Using monotone convergence, show that $f = \sum_{n=1}^{\infty} f_n \mathbb{1}_{X_n}$ satisfies the statement of the theorem.

To show uniqueness, consider another g which satisfies the statement of the theorem and show that $f = g$ a.e. (μ).

Corollary 5.8. Let $(X, \mathcal{S}, \mu)$ be a σ-finite measure space and ν a finite measure on $\mathcal{S}$. Then $\nu \ll \mu$ iff given any $\epsilon > 0$, $\exists \delta = \delta(\epsilon) > 0$ such that $|\nu(E)| < \epsilon$ whenever $E \in \mathcal{S}$ and $\mu(E) < \delta$.

Proof. For the first direction, note that if E is such that $\mu(E) = 0$, then $|\nu(E)| < \epsilon$ for all $\epsilon > 0$. For the other direction, apply a previous theorem regarding convergence of integrals.

5.3 Derivatives of measures

Definition. Let μ be a σ-finite measure and ν a σ-finite signed measure both on $(X, \mathcal{S})$ and such that $\nu \ll \mu$. The **Radon-Nikodym derivative of ν with respect to μ** is the function f appearing in the relation $\nu(E) = \int_E f \,\mathrm{d}\mu$.

Theorem 5.9. Let μ, ν be σ-finite measures in $(X, \mathcal{S})$ with $\nu \ll \mu$. If f is a measurable function defined on X and is non-negative or ν-integrable, then:

$$\int f \,\mathrm{d}\nu = \int f \left(\frac{\mathrm{d}\nu}{\mathrm{d}\mu} \right) \mathrm{d}\mu$$

Proof. Write $\mathrm{d}\nu/\mathrm{d}\mu = g$. For $E \in \mathcal{S}$, note that by Radon-Nikodym,

$$\int \mathbb{1}_E g \,\mathrm{d}\mu = \int_E g \,\mathrm{d}\mu = \nu(E) = \int \mathbb{1}_E \,\mathrm{d}\nu$$

So that the relation holds for $f = \mathbb{1}_E$. Thus the relation also holds for non-negative simple f. The full result then follows from monotone convergence.

Theorem 5.10. Let μ, ν be σ-finite measures on the measurable space $(X, \mathcal{S})$ and λ a σ-finite signed measure on $\mathcal{S}$. Then if $\lambda \ll \nu \ll \mu$,

$$\frac{\mathrm{d}\lambda}{\mathrm{d}\mu} = \frac{\mathrm{d}\lambda}{\mathrm{d}\nu} \cdot \frac{\mathrm{d}\nu}{\mathrm{d}\mu} \quad \text{a.e. } (\mu)$$

Proof. Assume λ is a measure (apply Jordan decomposition for signed measure). Let $f = \frac{\mathrm{d}\lambda}{\mathrm{d}\mu}$ and apply Radon-Nikodym.

5.4 Real line applications (READ HALMOS §43)

Definition. (Discrete measure)

Let $\mathcal{B}$ be the Borel sets of $\mathbb{R}$. A measure ν on $\mathcal{B}$ is **discrete** if there is a countable set C such that $\nu(C^c) = 0$.

> *Remark.* ($L_1(a,b)$ notation)
>
> From now on we introduce a new form of the L_1 notation. If $f \in L_1(a,b)$, then we mean that the absolute value of f is integrable on the interval (a,b).

Theorem 5.11. Let ν be a Lebesgue-Stieltjes measure on $\mathcal{B}$. Then there are uniquely determined measures ν_1, ν_2, ν_3 on $\mathcal{B}$ such that $\nu = \nu_1 + \nu_2 + \nu_3$ and $\nu_1 \ll m$, ν_2 is discrete, $\nu_3 \perp m$ with $\nu_3(\{x\}) = 0$ for all $x \in \mathbb{R}$.

> *Proof.* Apply Lebesgue decomposition to ν to obtain $\nu = \nu_0 + \nu_1$ where $\nu_1 \ll m$ and thus $\nu_0 \perp m$. Define $C = \{c : \nu_0(\{x\}) > 0\}$ and show C is countable.
>
> Define $\nu_2(B) = \nu_0(B \cap C)$ and $\nu_3(B) = \nu_0(B \cap C^c)$. The rest of the proof follows easily.

Definition. Given the setup above,

- ν_1 is the **absolutely continuous** part of ν.
- ν_2 is the **discrete singular** part of ν.
- ν_3 is the **continuous singular** part of ν.

Definition. Let F be a non-decreasing right continuous funtion defined on $\mathbb{R}$ and let μ_F be its corresponding Lebesgue-Stieltjes measure.

- If $\mu_F \ll m$, then F is **absolutely continuous** with density function $f = \mathrm{d}\mu_F/\mathrm{d}m$ (i.e. the Radon-Nikodym derivative of μ_F with respect to m).
- If F is continuous and $\mu_F \perp m$, then F is **singular**.

  F is *continuous* iff $\mu_F(\{x\}) = 0$ for all $x \in \mathbb{R}$.
- If μ_F is discrete, then F is **discrete**.

Theorem 5.12. Let F be a non-decreasing right continuous funtion defined on $\mathbb{R}$ and let μ_F be its corresponding Lebesgue-Stieltjes measure. If $\mu_F \ll m$, then F has density $f = \mathrm{d}\mu_F/\mathrm{d}m$, $f \in L_1(a,b)$ for $-\infty < a < b < \infty$ and

$$F(b) - F(a) = \mu_F\{(a,b]\} = \int_a^b f(t)\,\mathrm{d}t$$

And therefore, for each a and all x,

$$F(x) = F(a) + \int_a^x f(t)\,\mathrm{d}t \quad \textit{or, equivalently}$$
$$= F(a) + \int_{(a,x)} f\,\mathrm{d}m$$

Proof. Use the Radon-Nikodym theorem and the definition of Lebesgue-Stieltjes measure.

□

Corollary 5.13. Every non-decreasing, right-continuous function F defined on $\mathbb{R}$ has a decomposition

$$F(x) = F_1(x) + F_2(x) + F_3(x), \quad x \in \mathbb{R}$$

where F_1, F_2, F_3 are non-decreasing and right-continuous as well with F_1 absolutely continuous, F_2 discrete, and F_3 singular. F_1, F_2, and F_3 are unique up to an additive constant and F has at most countably many discontinuities arising solely from jumps in the discrete component F_2.

Proof. Let $\mu_F = \nu_1 + \nu_2 + \nu_3$ be the decomposition of μ_F into three components. Write $F_i(x) = \nu_i\{(0, x]\}$ for $x \geq 0$ and $-\nu_i\{[x, 0)\}$ for $x < 0$. Use the definition of Lebesgue-Stieltjes measure.

□

Theorem 5.14. Let F be an absolutely continuous function on $[a, b]$, $-\infty < a < b < +\infty$ with $F(x) = F(a) + \int_a^x f(t)\,\mathrm{d}t$, $f \in L_1(a, b)$ and g a Borel-measurable function defined on $\mathbb{R}$.

If $g(F(t))f(t) \in L_1(a, b)$ then $g(x) \in L_1(F(a), F(b))$ or $g(x) \in L_1(F(b), F(a))$ according as $F(a) < F(b)$ or $F(b) < F(a)$ respectively, and

$$\int_{F(a)}^{F(b)} g(x)\,\mathrm{d}x = \int_a^b g(F(t))f(t)\,\mathrm{d}t$$

where $\int_\alpha^\beta g(x)\,\mathrm{d}x = -\int_\beta^\alpha g(x)\,\mathrm{d}x$ for $\beta < \alpha$.

Proof. Hard.

□

Recall that we have previously defined absolute continuity of a function F in terms of the absolute continuity of the accompanying Lebesgue-Stieltjes measure μ_R. We now introduce a (slightly) more intuitive definition:

Definition. (Absolute continuity of functions)

A function F is **absolutely continuous on** $[a, b]$ if and only if for every $\epsilon > 0$, there is a $\delta = \delta(\epsilon) > 0$ such that

$$\sum_{i=1}^{n} |F(x_i') - F(x_i)| < \epsilon$$

for every finite collection $\{(x_i, x_i')\}_{i=1}^n$ of disjoint intervals in $[a, b]$ with $\sum_{i=1}^n |x_i' - x_i| < \delta$.

There are two basic facts about absolutely continuous functions:

1. All absolutely continuous functions are differentiable.
2. Absolute continuity is a stronger condition than continuity and uniform continuity.

We formalize the first in the next theorem (presented without proof).

Theorem 5.15. Every absolutely continuous function is differentiable $a.e.(m)$. In particular if F is absolutely continuous on $[a, b]$ and $F(x) = F(a) + \int_a^x f(t)\,\mathrm{d}t$, $a \le x \le b$, $f \in L_1(a, b)$, then $F'(x) = f(x)$ $a.e.(m)$ on $[a, b]$.

Moreover, if f is continuous, then $F'(x) = f(x)$ for all $a \le x \le b$.

We also present a new definition, useful in characterizing certain functions, and an accompanying existence theorem:

Definition. (Functions of bounded variation)

Let F be a real-valued function defined on $[a, b]$, $-\infty < a < b < \infty$. F is of **bounded variation** if it is the difference of two non-decreasing functions defined on $[a, b]$.

Since non-decreasing functions have at most a countable number of points of discontinuity (which must be jumps), the same is true for functions of bounded variation. Then if F is of bounded variation and is right-continuous, then $F = F_1 - F_2$, where F_1 and F_2 are non-decreasing and may be taken to be both right-continuous.

Theorem 5.16. (Existence of functions of bounded variation)

1. If F is a right-continuous function of bounded variation on $[a, b]$, $-\infty < a < b < +\infty$, then there exists a unique finite signed measure μ_F on the Borel subsets of $(a, b]$ such that $\mu_F\{(x, y]\} = F(y) - F(x)$ whenever $a \le x < y \le b$.
2. Conversely, if ν is a finite signed measure on the Borel subsets of $(a, b]$, $-\infty < a < b < +\infty$, then there exists a right-continuous function F of bounded variation on $[a, b]$ such that $\nu = \mu_F$. F is unique up to an additive constant.

   Proof. (both cases)

   1. As just noted above, we can decompose $F = F_1 - F_2$. Form the accompanying Lebesgue-Stieltjes measures of F_1 and F_2 and define the new measure $\mu_F = \mu_{F_1} - \mu_{F_2}$.

      Show uniqueness by using the identification lemma on the semiring of intervals $(x, y]$.

2. Form the Jordan decomposition $\nu = \nu_+ - \nu_-$. Define $F_1(x) = \nu_+(a, x]$ and $F_2(x) = \nu_-(a, x]$, $a \le x \le b$.

□

We close with some last connections between Lebesgue-Stieltjes measure, functions of bounded variation, and absolute continuity:

Definition. (Lebesgue-Stieltjes integral on $(a, b]$)

Let F be a right-continuous function of bounded variation on $[a, b]$, and let g be a Borel-measurable function such that the integral $\int_{(a,b])} g \, \mathrm{d}\mu_F$ is defined. Then the **Lebesgue-Stieltjes integral of** g **on** $(a, b]$ is defined as

$$\int_{(a,b]} g(x) \, \mathrm{d}F(x) = \int_{(a,b]} g \, \mathrm{d}F = \int_{(a,b]} g \, \mathrm{d}\mu_F$$

Theorem 5.17. Let F be a real-valued function on $[a, b]$. Then F is of bounded variation if and only if

$$\sup_{P \in \mathcal{P}} \sum_{n=1}^{N_P} |F(x_n) - F(x_n - 1)| < \infty$$

where $\mathcal{P} = \{P : P = \{x_1, x_2, \ldots, x_{N_P}\} \mid P \text{ is a partition of } [a, b]\}$.

Proof. HW.

□

Definition. (Total variation of a function)

The quantity $\sup_{P \in \mathcal{P}} \sum_{n=1}^{N_P} |F(x_n) - F(x_n - 1)|$ is called the **total variation** of F on $[a, b]$.

6 Convergence of measurable functions

6.1 Pointwise convergence concepts

Theorem 6.1. (a.u. $\to$ a.e.)

If $\{f_n\}$ is a sequence of functions $E \in \mathcal{S} \to \mathbb{R}^*$, and f_n is a.u. Cauchy ($f_n \to f$ a.u. on E) on E, then f_n is a.e. Cauchy on E ($f_n \to f$ a.e. on E).

Proof. If f is a.u. Cauchy on E, then $\exists\, F_p$ such that $\mu(F_p) < 1/p$ and $\{f_n\}$ is uniformly Cauchy (and thus Cauchy pointwise) on $E \setminus F_p$.

Consider $F = \cap_{p=1}^{\infty} F_p$. Note $\mu(F) = 0$ and if $x \in E - F$, then $x \in E - F_p$ for some p.

□

Theorem 6.2. (a.u. Cauchy $\to$ a.u.)

If $\{f_n\}$ is a.u. Cauchy on $E \in \mathcal{S}$, then there exists a measurable (?) function f such that $f_n \to f$ a.u. on E.

Proof. (see **Bartle, Lemma 7.10** for alternate method)

Note: we cannot just fix $\epsilon > 0$, exclude some set F_ϵ with $\mu(F_\epsilon) < \epsilon$ and argue that f_n is uniformly Cauchy on $E - F_\epsilon$ so that there exists some a.u. limiting function on $E - F_\epsilon$. That function may be different for every ϵ.

First, note that $\{f_n\}$ is Cauchy a.e. So there exists some function f such that $f_n \to f$ a.e. on E.

But also, since $\{f_n\}$ is a.u. Cauchy, it converges uniformly to a function g on $E - F_\epsilon$ for F_ϵ with $\mu(F_\epsilon) < \epsilon$. But by uniqueness of limits, $f = g$ on $E - F_\epsilon$.

□

Egoroff's Theorem. Let $\{f_n\}$ and f be measurable functions defined a.e. on $E \in \mathcal{S}$ such that $f_n \to f$ a.e. on E.

If the $\{f_n\}$ and f are finite and $\mu(E) < \infty$, then $f_n \to f$ a.u. as well.

Proof. (**Bartle, Theorem 7.12**)

Assume WLOG that f_n, f are defined, finite, and $f_n \to f$ on all of E. Define:

$$E_n(m) = \cap_{i=n}^{\infty} \{x \in E : |f_i(x) - f(x)| < 1/m\}$$

Argue that $\mu(E - E_n(m)) \to 0$ as $n \to \infty$. Thus given $\epsilon > 0$, there exists $N_m = N_m(\epsilon)$ (**Important:** depends on both m and ϵ) such that $\mu(E - E_n(m)) < \epsilon/2^m$ for $n \geq N_m$.

Define $F = F_\epsilon = \cup_{m=1}^{\infty} (E - E_{N_m}(m))$ and use F to show a.u. convergence.

□

6.2 Convergence in measure

Theorem 6.3. (In measure $\to$ Cauchy in measure)

If $f_n \to f$ in measure on $E \in \mathcal{S}$, then $\{f_n\}$ is Cauchy in measure on E.

Proof. Triangle inequality. □

Theorem 6.4. (Uniqueness of "in measure" limit)

If $\{f_n\}$ converges in measure on $E \in \mathcal{S}$ to both f and g, then $f = g$ a.e.

Proof. Triangle inequality. □

Theorem 6.5. (a.u. $\to$ in measure)

Let $\{f_n\}, f$ be measurable functions defined on $E \in \mathcal{S}$ and **finite a.e.**

If $\{f_n\}$ is Cauchy a.u. on E ($f_n \to f$ a.u. on E), then it is Cauchy in measure on E ($f_n \to f$ in measure on E).

Proof. Assume $\{f_n\}$ is Cauchy a.u. (the proof for ordinary a.u. is almost the same). Given any $\delta > 0$, there exists some set $F_\delta \subset E$ such that $f_n - f_m \to 0$ uniformly on $E - F_\delta$ as $n, m \to \infty$.

Now the rate at which $f_n - f_m \to 0$ uniformly **may depend on the set F_δ we are excluding**. Therefore, for any $\epsilon > 0$, there exists $N = N(\epsilon, \boldsymbol{\delta})$ such that $|f_n(x) - f_m(x)| < \epsilon$ for all $n, m \geq N$ and $x \in E - F_\delta$.

Fill in the details. □

Corollary 6.6. (a.e. $\to$ in measure)

Let $\{f_n\}, f$ be measurable functions defined on $E \in \mathcal{S}$ and **finite a.e.**

If $\mu(E) < \infty$ and $f_n \to f$ a.e., then $f_n \to f$ in measure on E.

Proof. By Egoroff. □

Theorem 6.7. (Cauchy in measure $\to$ Cauchy a.u.)

Let $\{f_n\}$ be a sequence of measurable functions defined on $E \in \mathcal{S}$ which is Cauchy in measure on E. Then there exists a subsequence $\{f_{n_k}\}$ which is Cauchy a.u. on E.

Proof. Three steps:

1. **Use Cauchyness in measure to define a subsequence:**

   Since $\{f_n\}$ is Cauchy in measure, then for any $k \in \mathbb{N}$ there exists n_k

such that

$$\mu\{x : f_{n_k}(x) - f_{n_{k+1}}(x)| \geq 2^{-k}\} \leq 2^{-k}$$

Arrange them $n_1 < n_2 < n_3 < \ldots$.

2. **Construct a set of measure** $< \epsilon$**:**

   Define:

   $$E_k = \{x : |f_{n_k}(x) - f_{n_{k+1}}(x)| \geq 2^{-k}\} \quad \text{and} \quad F_k = \cup_{m=k}^{\infty} E_m$$

   By a union bound $\mu(F_k) \leq 2^{-k+1}$. Now fix $\epsilon > 0$. Set k such that $2^{-k+1} < \epsilon$.

3. **Show a.u. convergence on** $E - F_k$**:**

   Note that if $x \in E - F_k$, then $x \in E - E_m$ for any $m \geq k$. Therefore:

   $$|f_{n_m}(x) - f_{n_{m+1}}(x)| < 2^{-m} \quad \text{for all } m \geq k$$

   It follows that for all $\ell \geq m \geq k$:

   $$|f_{n_m}(x) - f_{n_\ell}(x)| < 2^{-m+1} \to 0 \quad \text{as } m \to \infty$$

□

Theorem 6.8. (Cauchy in measure → in measure)

If $\{f_n\}$ is Cauchy in measure on E, then there exists an a.e. unique measurable function f defined on E such that $f_n \to f$ in measure on E.

Proof. By the previous result, there exists a subsequence $\{f_{n_k}\}$ which is Cauchy a.u. Since $f_n \to f$, then $f_{n_k} \to f$ as well.

Apply the triangle inequality to $|f_k(x) - f(x)|$. Show convergence as $k \to \infty$.

□

Corollary 6.9. (in measure → a.u.)

If $f_n \to f$ in measure on $E \in \mathcal{S}$, then there exists a subsequence $\{f_{n_k}\}$ such that $f_{n_k} \to f$ a.u. (and hence a.e.).

Proof. If $f_n \to f$ in measure on E, then it is Cauchy in measure on E. $\{f_n\}$ is Cauchy in measure on E, then there exists a subsequence $\{f_{n_k}\}$ which is Cauchy a.u. on E.

If $\{f_{n_k}\}$ is Cauchy a.u. on E, then there exists g such that $f_{n_k} \to g$ a.u. on E. Then $f_{n_k} \to g$ in measure on E also. But limits in measure are unique a.e.

□

Theorem 6.10. Let $\{f_n\}, f$ be measurable functions defined on $E \in \mathcal{S}$ and **finite a.e.**, with $\mu(E) < \infty$.

For any $\epsilon > 0$ and $n \in \mathbb{N}$, define $E_n(\epsilon) = \{x : |f_n(x) - f(x)| \geq \epsilon\}$. Then $f_n \to f$ a.e. if and only if for every $\epsilon > 0$,

$$\lim_{n\to\infty} \mu\{\cup_{m=n}^{\infty} E_m(\epsilon)\} = 0$$

Remark. i.e. $X_n \xrightarrow{a.s.} X$ iff for all $\epsilon > 0$, $\mathbb{P}(|X_n - X| \geq \epsilon \text{ i.o.}) = 0$.

Proof.

6.3 Basic L_p space results

Theorem 6.11. (Normed linear spaces are metric spaces)

Any normed linear space is a metric space with metric defined by:

$$d(f, g) = \|f - g\|$$

Proof.

Theorem 6.12. (Two results re: metric spaces)

Let (L, d) be a metric space and $\{f_n\}, \{g_n\}, f, g$ be elements of L. Then

1. If $f_n \to f$ and $f_n \to g$, then $f = g$.
2. If $f_n \to f$ and $g_n \to g$, then $d(f_n, g_n) \to d(f, g)$.

Proof. For (1), apply the triangle inequality.

For (2), apply the triangle inequality twice to obtain:

$$\begin{aligned} d(f_n, g_n) &\leq d(f_n, f) + d(f, g) + d(g_n, g) \\ d(f, g) &\leq d(f, f_n) + d(f_n, g_n) + d(g_n, g) \end{aligned}$$

Conclude.

□

Theorem 6.13. (L_p spaces are linear spaces)

For any $0 < p \leq \infty$, L_p is a linear space.

Remark. In particular, if $f, g \in L_p$ and $a, b \in \mathbb{R}$, then $af + bg \in L_p$ also.

Proof. Everything is trivial except showing closure under addition. Let $f, g \in L_p$. For $p = \infty$, the proof is obvious. For $0 < p < \infty$, note:

$$|f(x) + g(x)| \leq 2 \max\{|f(x)|, |g(x)|\}$$

Therefore:

$$\begin{aligned} |f(x) + g(x)|^p &\leq 2^p \max\{|f(x)|^p, |g(x)|^p\} \\ &\leq 2^p (|f(x)|^p + |g(x)|^p) \end{aligned}$$

Conclude.

□

Holder's Inequality. Let p, q be such that $1 \le p \le q \le \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$ (with $q = \infty$ when $p = 1$). If $f \in L_p$ and $g \in L_q$ then $fg \in L_1$ and

$$\|fg\|_1 \;\le\; \|f\|_p \, \|g\|_q$$

Proof. (**Bartle, Theorem 6.9**)

The case for either of $p, q = \infty$ is immediate from the fact that $\int |fg| \, \mathrm{d}\mu \le \|g\|_\infty \int |f| \, \mathrm{d}\mu$. To prove for the case of $1 < p, q < \infty$,

1. **Prove Young's inequality:**

   Either using the function $\phi(t) = \alpha t - t^\alpha$ for $0 < \alpha < 1$ or convexity of $\exp(\cdot)$ prove that:

   $$ab \le \frac{a^p}{p} + \frac{b^q}{q}, \quad a, b \ge 0$$

   for p, q such that $1/p + 1/q = 1$

2. **Plug in stuff:**

   Set $a = |f(x)|/\|f\|_p$ and $b = |g(x)|/\|g\|_q$. Take integrals of both sides.

□

Minkowski's Inequality. If $1 \le p \le \infty$ and $f, g \in L_p$, then $f + g \in L_p$ also and:

$$\|f + g\|_p \;\le\; \|f\|_p + \|g\|_p$$

Remark. The reverse inequality holds if $0 < p < 1$.

Proof. (**Bartle, Theorem 6.11**)

Assume $1 < p < \infty$.

1. **Show an easy inequality:**

   $$|f + g|^p \le |f| \cdot |f + g|^{p-1} + |g| \cdot |f + g|^{p-1}$$

2. **Argue that Holder applies:**

   Argue that there exists $q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$, and so $(p-1)q = p$. Use this fact to show that $|f + g|^{p-1}$ is L_q-integrable.

3. **Apply Holder**

   Apply Holder's inequality to $\int |f||f + g|^{p-1} \, \mathrm{d}\mu$ and $\int |g||f + g|^{p-1} \, \mathrm{d}\mu$ and combine with (1) to obtain:

   $$\|f + g\|_p^p \le (\|f\|_p + \|g\|_p) \cdot \|f + g\|_p^{p/q}$$

   Divide both sides by $\|f + g\|_p^{p-1}$ and use $p - p/q = 1$.

□

Theorem 6.14. Let $0 < p < 1$ and $f, g \in L_p$. Then:

$$\|f + g\|_p^p \leq \|f\|_p^p + \|g\|_p^p$$

Remark. Again, compare this to the reverse Minkowski inequality.

Proof. Since $0 < p < 1$, for any $t \geq 0$ we have $(1 + t)^p \leq 1 + t^p$. Set $t = a/b$ to get:

$$(a + b)^p \leq a^p + b^p \quad \text{any } a \geq 0,\, b > 0$$

Note $|f + g|^p \leq (|f| + |g|)^p$.

□

6.4 L_p completeness theorem

Theorem 6.15. (L_p spaces as normed linear/metric spaces)

1. If $1 \leq p \leq \infty$, then L_p is a normed linear space with norm $\|f\|_p$.
2. If $0 < p < 1$, then L_p is a metric space with metric $d_p(f, g) = \|f - g\|_p^p$.

Proof.

Theorem 6.16. (Cauchy in $L_p \to$ Cauchy in measure)

Let $0 < p \leq \infty$ and $\{f_n\}, f \in L_p$. If $\{f_n\}$ is Cauchy in L_p, then:

1. For $p < \infty$, $\{f_n\}$ is Cauchy in measure.
2. For $p = \infty$, $\{f_n\}$ is uniformly Cauchy a.e. (hence also Cauchy a.u. and Cauchy in measure).

Proof.

Theorem 6.17. (in $L_p \to$ in measure)

Let $0 < p \leq \infty$ and $\{f_n\}, f \in L_p$. If $f_n \to f$ in L_p, then:

1. For $p < \infty$, $f_n \to f$ in measure.
2. For $p = \infty$, $f_n \to f$ uniformly a.e. (hence also a.u. and in measure).

Furthermore, $\|f_n\|_p \to \|f\|_p$ and for $0 < p < \infty$,

$$\int |f_n|^p \, \mathrm{d}\mu \to \int |f|^p \, \mathrm{d}\mu$$

Proof.

Theorem 6.18. (L_p completeness theorem)

1. If $1 \leq p \leq \infty$, then L_p is a **Banach space** with norm $\|f\|_p$.
2. If $0 < p < 1$, then L_p is a **complete metric space** with metric $d_p(f, g) = \|f - g\|_p^p$.

Proof. (**Dudley, Theorem 5.2.1**)

Theorem 6.19. (Ordering of L_p spaces)

Let $(X, \mathcal{S}, \mu)$ be a finite measure space and let $0 < q \leq p \leq \infty$. Then:

1. $L_p \subset L_q$
2. For $f \in L_p$, $\|f\|_q \leq \|f\|_p (\mu(X))^{\frac{1}{q} - \frac{1}{p}}$.

Proof.

7 Product spaces

7.1 Measurability in Cartesian products

Lemma 7.1. (Rectangles from semirings are semirings)

If $\mathcal{S}, \mathcal{T}$ are semirings in X, Y respectively, then the class $\mathcal{P}$ of all rectangles $A \times B$ such that $A \in \mathcal{S}$ and $B \in \mathcal{T}$ is a semiring in the space $X \times Y$.

Proof.

Lemma 7.2. (Some properties of sections)

Let $X \times Y$ be the space formed by the Cartesian product of a space X and a space Y, and let $x \in X$.

1. If E, F are subsets of $X \times Y$, then $(E - F)_x = E_x - F_x$.
2. If $\{E_i\}_{i \geq 1}$ are subsets of $X \times Y$, then $(\cup_{i=1}^{\infty} E_i)_x = \cup_{i=1}^{\infty} (E_i)_x$ and $(\cap_{i=1}^{\infty} E_i)_x = \cap_{i=1}^{\infty} (E_i)_x$.

Proof.

Theorem 7.3. (Measurability of section-transformations)

Let $(X, \mathcal{S})$ and $(Y, \mathcal{T})$ be measurable spaces. Then the transformations:

1. $T_x : (Y, \mathcal{T}) \to (X \times Y, \mathcal{S} \times \mathcal{T})$ defined by $T_x(y) = (x, y)$
2. $T^y : (X, \mathcal{S}) \to (X \times Y, \mathcal{S} \times \mathcal{T})$ defined by $T^y(x) = (x, y)$

for $x \in X$, $y \in Y$, are measurable.

Proof.

Theorem 7.4. Let $(X, \mathcal{S})$ and $(Y, \mathcal{T})$ be measurable spaces. Let f be an $\mathcal{S} \times \mathcal{T}$-measurable function defined on a subset of $X \times Y$. Then every x-section f_x is $\mathcal{T}$-measurable and every y-section f^y is $\mathcal{S}$-measurable.

Proof.

7.2 Mixtures and integration with mixtures

Theorem 7.5. (Mixtures are measures)

Let $(X, \mathcal{S}, \mu)$ be a measure space and $(W, \mathcal{W})$ a measurable space.

Suppose that for every $x \in X$, λ_x is a measure on $\mathcal{W}$ such that for any fixed $E \in \mathcal{S}$, $\lambda_x(E)$ is $\mathcal{S}$-measurable in x. Define:

$$\lambda(E) = \int_X \lambda_x(E) \, \mathrm{d}\mu(x)$$

Then λ is a measure on $\mathcal{W}$, and $\lambda(E) = 0$ iff $\lambda_x(E) = 0$ a.e.(μ).

Proof.

Theorem 7.6. (Integration with mixture measures: basic)

Let $(X, \mathcal{S}, \mu)$ be a measure space and $(W, \mathcal{W})$ a measurable space.
Let λ_x be a measure on $\mathcal{W}$ for each $x \in X$ such that $\lambda_x(E)$ is $\mathcal{S}$-measurable as a function of x for each $E \in \mathcal{W}$, and let λ be the mixture measure on $\mathcal{W}$ (of the λ_x) defined previously.

Let f be a non-negative, $\mathcal{W}$-measurable function defined on **all of** W.

Then $\int_W f \, \mathrm{d}\lambda_x$ is a non-negative, $\mathcal{S}$-measurable function of x and:

$$\int_X \left\{ \int_W f \, \mathrm{d}\lambda_x \right\} \mathrm{d}\mu(x) = \int_W f \, \mathrm{d}\lambda$$

Proof.

Theorem 7.7. (Integration with mixture measures: generalized)

Let $(X, \mathcal{S}, \mu)$ be a measure space and $(W, \mathcal{W})$ a measurable space.
Let λ_x be a measure on $\mathcal{W}$ for each $x \in X$ such that $\lambda_x(E)$ is $\mathcal{S}$-measurable as a function of x for each $E \in \mathcal{W}$, and let λ be the mixture measure on $\mathcal{W}$ (of the λ_x) defined previously.

Let f be a non-negative, $\mathcal{W}$-measurable function defined **a.e.**(λ) on W.

1. (Case 1: non-negative f)

   If f is non-negative a.e.(λ), then $\int_W f \, \mathrm{d}\lambda_x$ is a non-negative, $\mathcal{S}$-measurable function of x **defined a.e.(μ) on** X and:

   $$\int_X \left\{ \int_W f \, \mathrm{d}\lambda_x \right\} \mathrm{d}\mu(x) = \int_W f \, \mathrm{d}\lambda$$

2. (Case 2: integrable f)

   If $\int_W |f| \, \mathrm{d}\lambda < \infty$ or $\int_X \{\int_W |f| \, \mathrm{d}\lambda_x\} \, \mathrm{d}\mu(x) < \infty$, then $f \in L_1(W, \mathcal{W}, \lambda_x)$ and $\int_W f \, \mathrm{d}\lambda_x \in L_1(X, \mathcal{S}, \mu)$ and:

   $$\int_X \left\{ \int_W f \, \mathrm{d}\lambda_x \right\} \mathrm{d}\mu(x) = \int_W f \, \mathrm{d}\lambda$$

7.3 "Generalized" product measure

Theorem 7.8. (Existence of "generalized" product measure)

Let $(X, \mathcal{S}, \mu)$ be a measure space and $(W, \mathcal{W})$ a measurable space.
Let ν_x be a measure on $\mathcal{T}$ for each $x \in X$.

Suppose that $\nu_x(B)$ is $\mathcal{S}$-measurable in x for each $B \in \mathcal{T}$ and $\{\nu_x : x \in X\}$ is uniformly σ-finite. Then:

1. $\nu_x(E_x)$ is $\mathcal{S}$-measurable for each $E \in \mathcal{S} \times \mathcal{T}$.

2. Define λ on $\mathcal{S} \times \mathcal{T}$ by $\lambda(E) = \int_X \nu_x(E_x)\,\mathrm{d}\mu(x)$. Then λ is a **measure** satisfying:

$$\lambda(A \times B) = \int_A \nu_x(B)\,\mathrm{d}\mu(x) \quad \text{for } A \in \mathcal{S}, B \in \mathcal{T}$$

3. If also $\int_{A_n} \nu_x(B_m)\,\mathrm{d}\mu(x) < \infty$, $m, n = 1, 2, \ldots$ for some sequence of sets $A_n \in \mathcal{S}$ with $\cup_1^\infty A_n = X$, then λ is the **unique** measure on $\mathcal{S} \times \mathcal{T}$ with the above property.

   Remark. ν_x is a measure on $(Y, \mathcal{T})$. λ is a measure on $(X \times Y, \mathcal{S} \times \mathcal{T})$.

   Proof.

Theorem 7.9. (Integration with "generalized" product measure)

Let $(X, \mathcal{S}, \mu)$ be a measure space and $(W, \mathcal{W})$ a measurable space.
Let ν_x be a measure on $\mathcal{T}$ for each $x \in X$.

Suppose that $\nu_x(B)$ is $\mathcal{S}$-measurable in x for each $B \in \mathcal{T}$ and $\{\nu_x : x \in X\}$ is uniformly σ-finite. Recall the "generalized" product measure λ on $\mathcal{S} \times \mathcal{T}$:

$$\lambda(E) = \int_X \nu_x(E_x)\,\mathrm{d}\mu(x)$$

Let f be a measurable function defined a.e.(λ) on $\mathcal{S} \times \mathcal{T}$.

1. (Case 1: non-negative f)

   If f is non-negative a.e.(λ), then $\int_Y f_x\,\mathrm{d}\nu_x$ is a non-negative, $\mathcal{S}$-measurable function of x **defined a.e.(μ) on** X and:

$$\int_X \left\{ \int_Y f_x\,\mathrm{d}\nu_x \right\} \mathrm{d}\mu(x) = \int_{X \times Y} f\,\mathrm{d}\lambda$$

2. (Case 2: integrable f)

   If $\int_{X \times Y} |f|\,\mathrm{d}\lambda < \infty$ or $\int_X \{\int_Y |f_x|\,\mathrm{d}\nu_x\}\,\mathrm{d}\mu(x) < \infty$, then $\int_Y f_x\,\mathrm{d}\nu_x \in L_1(X, \mathcal{S}, \mu)$ and:

$$\int_X \left\{ \int_Y f_x\,\mathrm{d}\nu_x \right\} \mathrm{d}\mu(x) = \int_{X \times Y} f\,\mathrm{d}\lambda$$

   Proof.

7.4 Fubini's theorem

Theorem 7.10. Let $(X, \mathcal{S}, \mu)$ be a measure space and $(Y, \mathcal{T}, \nu)$ a σ-finite measure space. Define λ by:

$$\lambda(E) = \int_X \nu(E_x)\,\mathrm{d}\mu(x) \quad \text{for } E \in \mathcal{S} \times \mathcal{T}$$

1. λ is a measure on $\mathcal{S} \times \mathcal{T}$ with $\lambda(A \times B) = \mu(A) \cdot \nu(B)$ for $A \in \mathcal{S}$, $B \in \mathcal{T}$.
2. If μ is σ-finite, then also:

   $$\lambda(E) = \int_Y \nu(E^y)\,\mathrm{d}\nu(y)$$

   and λ is σ-finite and is the unique measure on $\mathcal{S} \times \mathcal{T}$ satisfying the identity in (1).

   Proof.

Corollary 7.11. Let $(X, \mathcal{S}, \mu)$ and $(Y, \mathcal{T}, \nu)$ be σ-finite measure spaces.

Then for any fixed $E \in \mathcal{S} \times \mathcal{T}$, $(\mu \times \nu)(E) = 0$ iff $\nu(E_x) = 0$ a.e.(μ) iff $\mu(E^y) = 0$ a.e.(ν).

Proof.

Fubini's Theorem. Let $(X, \mathcal{S}, \mu)$ and $(Y, \mathcal{T}, \nu)$ be σ-finite measure spaces. Let f be an $\mathcal{S} \times \mathcal{T}$-measurable function defined a.e.$(\lambda = \mu \times \nu)$ on $X \times Y$. Then:

1. (Case 1: non-negative f)

   If $f \geq 0$ a.e.(λ), then $\int_Y f_x\,\mathrm{d}\nu$ is $\mathcal{S}$-measurable and $\int_X f^y\,\mathrm{d}\mu$ is $\mathcal{T}$-measurable and:

   $$\int_{X \times Y} f\,\mathrm{d}\lambda = \int_X \left\{ \int_Y f_x\,\mathrm{d}\nu \right\} \mathrm{d}\mu(x) = \int_Y \left\{ \int_X f^y\,\mathrm{d}\mu \right\} \mathrm{d}\nu(y)$$

2. (Case 2:)

   If one of the following three equivalent conditions hold:

   (a) $\int_{X \times Y} |f|\,\mathrm{d}\lambda < \infty$

   (b) $\int_X \{\int_Y |f_x|\,\mathrm{d}\nu\Big\}\,\mathrm{d}\mu(x) < \infty$

   (c) $\int_Y \{\int_X |f^y|\,\mathrm{d}\mu\}\,\mathrm{d}\nu(y) < \infty$

   Then:

   $$\int_{X \times Y} f\,\mathrm{d}\lambda = \int_X \left\{ \int_Y f_x\,\mathrm{d}\nu \right\} \mathrm{d}\mu(x) = \int_Y \left\{ \int_X f^y\,\mathrm{d}\mu \right\} \mathrm{d}\nu(y)$$

7.5 Signed measures on product spaces

7.6 Real line applications

7.7 Finite-dimensional product spaces

7.8 Lebesgue-Stieltjes measures on $\mathbb{R}^n$

7.9 The space $(\mathbb{R}^T, \mathcal{B}^T)$

7.10 Measures on $\mathbb{R}^T$, Kolmogorov's extension theorem